MU, Study of a Fundamental Spectrum in Homotopy Theory

Master thesis defence

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What is MU ?

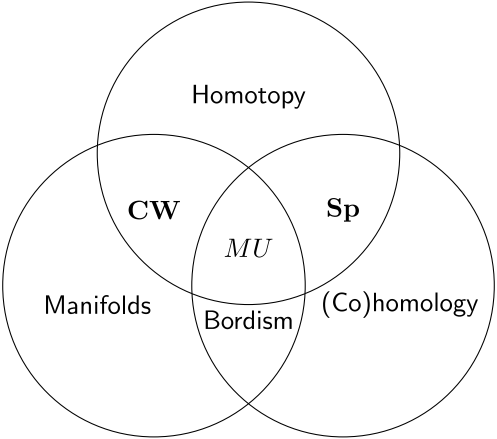
MU is *simply* the spectrum (1) induced by the Thomification (3) of the universal complex vector bundles (2).

Definition (MU spectrum)

Let $n \in \mathbb{Z}$. MU is given by:

- $(MU)_n = \begin{cases} * & n < 0 \\ MU(k) = T(\gamma_k^{\mathbb{C}}) & n = 2k \\ \Sigma MU(k) & n = 2k+1 \end{cases}$
- $p_{2n} = id_{\Sigma MU(n)}$
  $p_{2n+1} = T(j) : \Sigma^2 MU(n) = T(j^*(\gamma_{n+1})) \hookrightarrow MU(n+1)$.

Why MU ?



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CW-complexes

Definition (CW category)

$$Ob(\mathbf{CW}) = \left\{ X |\ X \text{ is a CW-complex } \right\}$$

$$\mathbf{CW}(X, Y) = \{ f : X \to Y \text{ continuous } \}$$

Definition (naive homotopy HCW category)

$$Ob(\mathbf{HCW}) = Ob(\mathbf{CW})$$

$$\mathbf{HCW}(X, Y) = [X, Y]$$

$$\mathbf{HCW} = \mathbf{HoCW} \cong \mathbf{HoTop_\bullet}$$

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Brown's Representability Theorem

Brown's Representability Theorem (1955)

For $F : \mathbf{HoCW} \to \mathbf{Set}$ or $\mathbf{Gp}$ any contravariant functor satisfying $\mathcal{W}$ and $\mathcal{MV}$.

- $\exists Y \in Ob(\mathbf{CW})$
- $\exists u \in F(Y)$
- $\forall X \in Ob(\mathbf{CW})$

$$T_u : [X, Y] \cong F(X)$$

with $T_u(f) = F(f)(u)$.

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- $F = H^n(-)$ a reduced cohomology theory satifying $\mathcal{W}$ yields $\{E_n\}_{n\in\mathbb{Z}}$ such that:

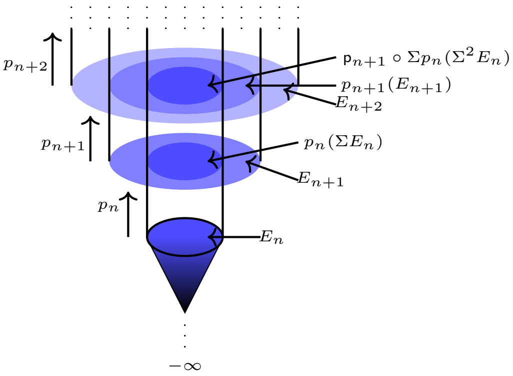
$$[X, \Omega E_{n+1}] \xrightarrow[\cong]{A^{-1}} [\Sigma X, E_{n+1}] \xrightarrow[\cong]{T_{n+1}} k^{n+1}(\Sigma X) \xrightarrow[\cong]{\sigma} k^n(X) \xrightarrow[\cong]{T_n^{-1}} [X, E_n]$$

$$E_n \sim_{Hom} \Omega E_{n+1}.$$

Spectrum

Definition (1 spectrum)

- A **spectrum** E is a collection $\{E_n\}_{n\in\mathbb{Z}}$ of CW complexes with injective maps
$$p_n : \Sigma E_n \hookrightarrow E_{n+1}.$$
- A **subspectrum** $F \subset E$ is a subcollection $F_n \subset E_n$ such that $p_n(\Sigma F_n) \subset F_{n+1}$.



Examples

$$\mathbb{S} = \begin{cases} * & n < 0 \\ S^n & n \geq 0 \end{cases}$$

Spectral functions

Definition (spectral functions)

Let E, F be spectra. A **function** $f : E \to F$ is a collection of maps $\{f_n : E_n \to F_n\}_{n\in\mathbb{Z}}$ such that:

$$f_{n+1}|_{p_n(\Sigma E_n)} = p'_n \circ \Sigma f_n$$

$$\begin{array}{ccc} E_{n+1} & \xrightarrow{\quad f_{n+1} \quad} & F_{n+1} \\ p_n \uparrow & & \uparrow p'_n \\ \Sigma E_n & \xrightarrow{\quad \Sigma f_n \quad} & \Sigma F_n \end{array}$$

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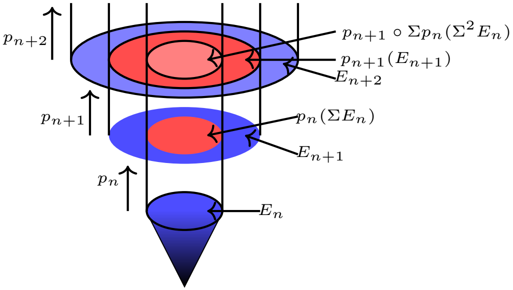
Cofinal subspectra

Definition (cofinal subspectra)

Let E be any spectrum, $F \subset E$ a subspectrum is **cofinal** if $\forall e_n \in E_n, \exists m$ such that

$$p_{n+m} \circ \Sigma p_{n+m-1} \circ \cdots \circ \Sigma^{m-1} p_n(\Sigma^m e_n) \in F_{n+m}.$$



Spectral maps

Definition (spectral maps)

Let E, F be spectra.

$$S = \Big\{(E', f')|\ E' \text{ is a cofinal subspectrum of E}, f' : E' \to F\Big\}$$

- $(E', f') \sim (E'', f'') \iff \exists(\widetilde{E}, \widetilde{f}) \in S$ s.t.

$$\widetilde{E} \subseteq E' \cap E''$$

$$\widetilde{f} = f'|_{\widetilde{E}} = f''|_{\widetilde{E}}$$

- The equivalence class $[E', f']$ is called a **map** from E into F

$$Hom(E, F) = S/_{\sim}$$

Spectra category $\mathbf{Sp}$

Definition $\mathbf{Sp}$

$$Ob(\mathbf{Sp}) = \{E | \ E \text{ is a spectrum}\}$$

$$\mathbf{Sp}(E,F) = Hom(E,F)$$

Definition (∞-suspension)

$$\Sigma^\infty : \mathbf{CW} \to \mathbf{Sp}$$

$$\Sigma^\infty X = \begin{cases} * & n < 0 \\ \Sigma^n X & n \geq 0 \end{cases}$$

$$\Sigma^\infty(f) = \begin{cases} id_* & n < 0 \\ \Sigma^n f & n \geq 0 \end{cases}$$

Definition (spectral suspension)

$$\Sigma : \mathbf{Sp} \to \mathbf{Sp}$$

$$(\Sigma E)_n = E_{n+1}$$

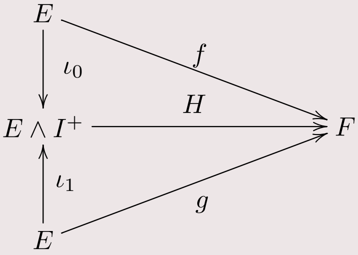
$$(\Sigma f)_n = f_{n+1}$$

Spectral Homotopy

Definition (homotopy)

Let $E, F \in Ob(\mathbf{Sp})$, $f, g \in \mathbf{Sp}(E, F)$.

- f is **homotopic** to g ($f \sim_{Hom} g$) if there exists a map $H : E \wedge I^+ \to F$ s.t.



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Spectral Homotopy

- $[E, F] = \mathbf{Sp}(E, F)/_{\sim_{Hom}}$
- $\pi_n(E) = [\Sigma^n \mathbb{S}, E]$ for $n \in \mathbb{Z}$
- $f \in \mathbf{Sp}(E, F)$ induces:
  - **pushforwards**

$$f_* : [G, E] \to [G, F]$$
$$f_*\big([h]\big) = [f \circ h]$$

  - **pullbacks**

$$f^* : [F, G] \to [E, G]$$
$$f^*\big([h]\big) = [h \circ f]$$

Spectral Homotopy

- $\pi_n(E) \cong \text{colim}_k\ \pi_{n+k}(E_k)$

Whitehead's Spectral Theorem

Let $E, F \in Ob(\mathbf{Sp})$, $f \in \mathbf{Sp}(E, F)$.

f is an homotopy equivalence $\iff$ f is a weak homotopy equivalence.

Definition ($\mathbf{HoSp} = \mathbf{HSp}$)

$$Ob(\mathbf{HoSp}) = Ob(\mathbf{Sp})$$

$$\mathbf{HoSp}(E, F) = [E, F]$$

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Reduced (co)homology induced by spectra

Definition (reduced homology induced by $E \in Ob(\mathbf{Sp})$)

$$E_* : \mathbf{HoCW} \to \mathbf{Ab}$$
$$E_n(X) = \pi_n(E \wedge X) = [\Sigma^n \mathbb{S}, E \wedge X]$$
$$E_n(f) = (id_E \wedge f)_*$$

Definition (reduced cohomology induced by $E \in Ob(\mathbf{Sp})$)

$$E^* : \mathbf{HoCW} \to \mathbf{Ab}$$
$$E^n(X) = [\Sigma^\infty X, \Sigma^n E]$$
$$E^n(f) = (\Sigma^\infty f)^*$$

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Back to Brown

Brown's Representability Theorem on cohomology

Let $k^* : \mathbf{HoCW} \to \mathbf{Ab}$ be any reduced cohomology satisfying $\mathcal{W}$.
There exist

- $E \in Ob(\mathbf{Sp})$
- A natural equivalence

$$T : E^*(-) \cong k^*(-)$$

Corollary

$$\mathbf{HoSp} \cong \mathbf{cohom}_S$$

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(1) (3) (2)

MU is *simply* the spectrum induced by the Thomification of the universal complex vector bundles.

Definition (MU spectrum)

Let $n \in \mathbb{Z}$. MU is given by:

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Vector bundles

Definition (Vector bundle)

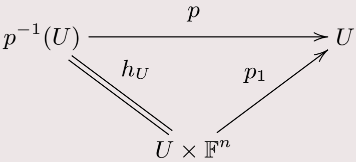
An $\mathbb{F}$-**vector bundle** of dimension n over B is $\xi = (E, B, p)$ with p a continuous mapping

$$p : E \to B$$

s.t. for all $b \in B$, there exist

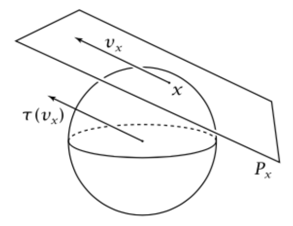
1. U an open neighbourhood of b,
2. $h_U : p^{-1}(U) \cong U \times \mathbb{F}^n$ an homeomorphism,

s.t.



Vector bundles

Examples



1. TM with M a manifold.
2. $N(M, W)$ with M embedded submanifold of W.
3. $(B \times \mathbb{F}^n, B, p_1)$ is the trivial bundle denoted $\epsilon_n^{\mathbb{F}}(B)$.

Vector bundles

Definition (Sum of vector bundles)

Let $\xi_1 = (E_1, X_1, p_1)$, $\xi_2 = (E_2, X_2, p_2)$ be vector bundles.

- The **external sum** $\xi_1 \times \xi_2$ is the vector bundle

$$(E_1 \times E_2, X_1 \times X_2, p_1 \times p_2).$$

Vector bundles

Definition (pullback of vector bundle)

Let $\xi = (E, Y, p)$ be any vector bundle and $f : X \xrightarrow{cont.} Y$.

- The **pullback of** ξ **by** f, $f^*(\xi)$ is the vector bundle (E_f, X, p_f) with:
  - $E_f = \Big\{(x, e) \in X \times E |\ f(x) = p(e)\Big\}$
  - $p_f(x, e) = x$

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Vector bundles

Lemma

Let $X, Y \in Ob(\mathbf{CW})$, $f, g : \mathbf{CW}(X, Y)$ and ξ be any vector bundle on Y.

$$f \sim_{Hom} g \implies f^*(\xi) \cong g^*(\xi).$$

The following functor satisfies $\mathcal{W}$ and $\mathcal{MV}$. Hence, Brown's representability theorem applies.

Definition (vector bundle contravariant functor)

$$Vb_n^{\mathbb{F}} : \mathbf{HoCW} \to \mathbf{Set}$$

1. $Vb_n^{\mathbb{F}}(X) = Iso\Big\{\xi \mid \xi \text{ is a } n\text{-dimensional } \mathbb{F}\text{-vector bundles on } X\Big\}$
2. $Vb_n^{\mathbb{F}}\big([f]\big)(\xi) = f^*(\xi)$.

Universal bundles

Corollary

Let $n \in \mathbb{N}$.

- $\exists BU(n) \in Ob(\mathbf{CW})$, $\exists u_n$ an n-complex vector bundle on $BU(n)$,
- $\forall X \in Ob(\mathbf{CW})$, $\forall \xi$ any n-complex vector bundle on X,
- $\exists f : X \to BU(n)$ s.t.

$$\xi \cong f^*(u_n)$$

(2)

$BU(n)$ is called the **classifying space** and u_n the **n-th universal bundle**.

Universal bundles

Corollary

Let $n \in \mathbb{N}$.

- $\exists BO(n) \in Ob(\mathbf{CW})$, $\exists o_n$ an n-real vector bundle on $BO(n)$,
- $\forall X \in Ob(\mathbf{CW})$, $\forall \xi$ any n-real vector bundle on X,
- $\exists f : X \to BO(n)$ s.t.

$$\xi \cong f^*(o_n)$$

Universal bundles

- We have proved that $BU(n)$ and u_n exist.
- We can in fact construct them.

Infinite Grassmannians

Definition (infinite Grassmannians)

$$G_n^{\mathbb{F}} = \left\{ K \subset \mathbb{F}^{\infty} \middle| \; K \text{ any linear subspace of dimension } n \; \right\}$$

Definition (tautological bundle γ_n)

$$\gamma_n^{\mathbb{F}} = (E, G_n^{\mathbb{F}}, p)$$

$$E = \{(K, v) \in G_n^{\mathbb{F}} \times \mathbb{F}^{\infty} | v \in K\}$$

$$p(K, v) = K$$

Universal Representation Theorem

$$BU(n) = G_n^{\mathbb{C}} \qquad BO(n) = G_n^{\mathbb{R}}$$

$$u_n \cong \gamma_n^{\mathbb{C}} \qquad o_n \cong \gamma_n^{\mathbb{R}}$$

(1) (3) (2)

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Disks/Sphere Bundles

- A Riemannian/Hermitian metric on the total space derives from any given $X \in Ob(\mathbf{CW})$ and vector bundle ξ on X.

Definition (disks/sphere bundles)

Let ξ be a complex vector bundle on B equipped with an Hermitian metric.

- The **disk bundle** $D(\xi)$ on B is

$$E_D = \Big\{(x, v) \in E| \ |v| \leqslant 1\Big\}$$

- The **sphere bundle** $S(\xi)$ on B is

$$E_S = \Big\{(x, v) \in E| \ |v| = 1\Big\}$$

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Thom Space

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Definition (Thom space)

Let ξ be a complex vector bundle over $B \in Ob(\mathbf{CW})$ equipped with an Hermitian metric.

- The **Thom space** of ξ is $T(\xi) \in Ob(\mathbf{CW})$:

$$T(\xi) = D(\xi)/S(\xi)$$

- For $f \in \mathbf{CW}(X, B)$, we define the **Thomification map**:

$$T(f) : T(f^*\xi) \to T(\xi)$$

Properties of Thom Space

- $T(\xi_1 \times \xi_2) \cong T(\xi_1) \wedge T(\xi_2)$

Properties of Thom Space

- $T(\xi_1 \times \xi_2) \cong T(\xi_1) \wedge T(\xi_2)$
- $T\Big(\epsilon_{\mathbb{C}}^n(X)\Big) \cong X^+ \wedge S_{\mathbb{R}}^{2n}$
- $T\Big(\xi_1 \oplus \epsilon_{\mathbb{C}}^n(X)\Big) \cong T(\xi_1) \wedge S_{\mathbb{R}}^{2n}$

Properties of Thom Space

- $T(\xi_1 \times \xi_2) \cong T(\xi_1) \wedge T(\xi_2)$
- $T\Big(\epsilon_{\mathbb{C}}^n(X)\Big) \cong X^+ \wedge S_{\mathbb{R}}^{2n}$
- $T\Big(\xi_1 \oplus \epsilon_{\mathbb{C}}^n(X)\Big) \cong T(\xi_1) \wedge S_{\mathbb{R}}^{2n}$
- Let X is any compact Hausdorff space, ξ any vector bundle on X.

  $$T(\xi) \cong E^\dagger$$

  with $E^\dagger$ the one point compactification of E.
- γ_1 is the universal bundle on $G_1^{\mathbb{C}} = \mathbb{C}P^\infty$

  $$T(\gamma_1) \cong \mathbb{C}P^\infty$$

MU spectrum

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- Using $j : G_{\mathbb{C}}^n \hookrightarrow G_{\mathbb{C}}^{n+1}$, we have that $j^*(\gamma_{n+1}^{\mathbb{C}}) \cong \gamma_n^{\mathbb{C}} \oplus \epsilon^{\mathbb{C}}$.

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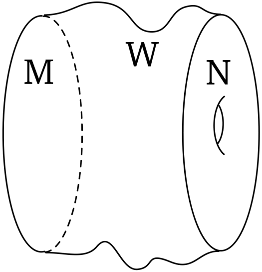
Example of bordism



Cobordism W between $M = S^2_{\mathbb{R}}$ and $N = (S^1_{\mathbb{R}} \times S^1_{\mathbb{R}})$

Thom-Pontrjagin Isomorphism

Thom-Pontrjagin Isomorphism (1959)

For $n \in \mathbb{Z}$, $X \in Ob(\mathbf{CW})$.

$$\Phi : \Omega_n^U(X) \cong \pi_n(MU \wedge X^+)$$

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Cohomology of MU

Cohomology of MU

The cohomology of MU can be entirely computed. For $k \in \mathbb{N}$:

- $H^{2k+1}(MU) = 0$
- $H^{2k}(MU) = H^{2k}(BU(k)) = \mathbb{Z}^{\alpha_k(k)}$ with

$$\alpha_n(k) = \begin{cases} \alpha_n(k-n) + \alpha_{n-1}(k) \\ 1 \text{ if } n = k = 0 \\ 0, \text{ if } k < 0 \text{ or } k \neq n = 0 \end{cases}$$

	H^0	H^2	H^4	H^6	H^8	H^{10}	H^{12}	H^{14}	H^{16}	H^{18}	H^{20}
$BU(0)$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
$BU(1)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$BU(2)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^5$	$\mathbb{Z}^6$
$BU(3)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{14}$
$BU(4)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^6$	$\mathbb{Z}^9$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{15}$	$\mathbb{Z}^{18}$	$\mathbb{Z}^{23}$
$BU(5)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{13}$	$\mathbb{Z}^{18}$	$\mathbb{Z}^{23}$	$\mathbb{Z}^{30}$
$BU(6)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{20}$	$\mathbb{Z}^{26}$	$\mathbb{Z}^{35}$
$BU(7)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{15}$	$\mathbb{Z}^{21}$	$\mathbb{Z}^{28}$	$\mathbb{Z}^{38}$
$BU(8)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{15}$	$\mathbb{Z}^{22}$	$\mathbb{Z}^{29}$	$\mathbb{Z}^{40}$

Nilpotence Theorem

Nilpotence Theorem (1980)

Let R be any ring spectrum. Consider the Hurewicz map

$$\pi_\bullet(R) \xrightarrow{h} MU_\bullet(R)$$

Then, $\alpha \in \pi_\bullet(R)$ is nilpotent to multiplication $\iff h(\alpha) = 0$.

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APPENDIX

$\mathbf{Top}_\bullet$ category

Definition $\mathbf{Top}_\bullet$

$$Ob(\mathbf{Top}_\bullet) = \Big\{(X, x_0)|\ X \text{ is a topological space and } x_0 \in X\Big\}$$

$$\mathbf{Top}_\bullet\big((X, y_0), (Y, y_0)\big) = \Big\{f : X \to Y|\ f \text{ continuous and } f(x_0) = y_0\Big\}$$

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Cones & Cofibrations

- **Cone** of (X, x_0)
$$CX = X \times [0,1]_{/\sim}.$$
with $(x,0) \sim (x_0, t)$
- **Mapping cone** of $f \in \mathbf{Top}_\bullet\big((X, y_0), (Y, y_0)\big)$
$$Y \cup_f CX = Y \cup CX_{/\sim}$$
with $(x,1) \sim f(x)$.
- **Cofibration** are sequence given by
$$X \xrightarrow{f} Y \xrightarrow{\iota} Y \cup_f CX$$

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Cell-complexes

Definition cells complexes

A **cell-complex** K is construct by induction on the n-**skeleton** K^n:

- $$K^{-1} = \{x_0\}$$

- $$K^0 = \bigcup_\alpha K^{-1} \cup_{x_0} S^0_\alpha$$

- $\forall n \in \mathbb{N}^+$, we consider a collection of map $\{f_\alpha : S^{n-1} \to K^{n-1}\}$.

$$\begin{aligned} K^n &= \bigcup_\alpha K^{n-1} \cup_{f_\alpha} CS^{n-1} \\ &= \bigcup_\alpha K^{n-1} \cup_{f_\alpha} D^n \end{aligned}$$

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Definition CW complexes

A **CW-complex** is a K cell-complex such that

- *C)* K is **closure-finite**. i.e:

$$(e_\alpha^n \cap e_\beta^m) \backslash x_0 = \emptyset \text{ except on finitely many occasions }.$$

- *W)* It has the **weak topology** induced by K^n. i.e:

$$S \subseteq K \text{ is closed } \iff \forall n \in \mathbb{N}, \alpha \in J_n, S \cap e_\alpha^n \text{ is closed in } e_\alpha^n.$$

Definition cellular maps

Let X, Y be CW-complexes, $f : X \to Y$ a continuous map is said **cellular** if $\forall n \in \mathbb{N}\ f(X^n) \subset Y^n$.

Definition Quillen homotopy category

$$Ob(\mathbf{HoCW}) = Ob(\mathbf{CW})$$

$$\mathbf{HoCW}(X, Y) = [X, Y][\mathcal{W}^{-1}]$$

where $[X, Y][\mathcal{W}^{-1}]$ is the set $[X, Y]$ localised on the class $\mathcal{W}$ of all weak equivalences.

CW-Approximation theorem

$\forall (X, x_0) \in \mathbf{Top}_\bullet$

$\exists Y$ a CW-complex and $f : Y \to X$ a weak homotopy equivalence.

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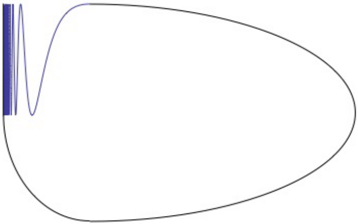
Example of WHE $\nRightarrow$ HE in $\mathbf{Top}$.



$$W = \Big\{\{0\} \times [-1,1]\Big\} \cup \Big\{\big(x, \sin(\frac{1}{x})\big) \mid x \in (0,t]\Big\}_{/\sim}$$

with $(t, \sin(\frac{1}{t})) \sim (0,-1)$.

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Homology

Definition reduced homology

A family $\{H_n : \mathbf{HCW} \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ of functors with $\{\sigma_n : H_n \to H_{n+1} \circ \Sigma\}_{n\in\mathbb{Z}}$ is called a **reduced homology theory** $H_*(-)$ if for all cofibration
$X \xrightarrow{f} Y \xrightarrow{j} Y \cup_f CX$,

$$H_n(A) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{H_n(j)} H_n(Y \cup_f CX)$$

is exact.

$Y \cup_f CX = Y \cup CX_{/\sim}$ with $(x, 1) \sim f(x)$

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Cohomology

Definition reduced cohomology

A family $\{H^n : \mathbf{HCW} \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ of contravariant functors and natural equivalences $\{\sigma^n : H^{n+1} \circ \Sigma \to H^n\}_{n\in\mathbb{Z}}$ is called a **reduced cohomology theory** H^* if for all cofibration $X \xrightarrow{f} Y \xrightarrow{j} Y \cup_f CX$,

$$H^n(X) \xleftarrow{H^n(f)} H^n(Y) \xleftarrow{H^n(j)} H^n(Y \cup_f CX)$$

is exact.

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Axioms of contravariant functors

Let $F : \mathbf{HCW} \to \mathbf{Set}, \mathbf{Gp}, \cdots$

- **Wedge** $\mathcal{W}$: Using $i_\alpha : X_\alpha \hookrightarrow \bigvee_{\alpha \in A} X_\alpha$,

$$\big(F(i_\alpha)\big)_{\alpha \in A} : F(\bigvee_{\alpha \in A} X_\alpha) \cong \prod_{\alpha \in A} F(X_\alpha)$$

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- **Mayer-Vietoris** $\mathcal{MV}$: For any CW-triad (X, A_1, A_2) with $x_1 \in F(A_1), x_2 \in F(A_2)$ such that

$$F(i_{A_1 \cap A_2})(x_1) = x_1|_{A_1 \cap A_2} = x_2|_{A_1 \cap A_2} = F(i_{A_1 \cap A_2})(x_2)$$

  Then, $\exists y \in F(X)$ such that $y|_{A_1} = x_1, y|_{A_2} = x_2$.

  Any reduced cohomology $H^*(-)$ follows $\mathcal{MV}$ and sometimes $\mathcal{W}$.

Grassmannians

Definition (Grassmannians)

$$G^{\mathbb{F}}_{n,k} = \{K \subset \mathbb{F}^{n+k} | \ K \text{ linear subspace of dimension } n \ \}$$

$$A \subset G^{\mathbb{F}}_{n,k} \text{ is open } \iff A = \{K | \ K \subset U, U \text{ open in } \mathbb{F}^{n+k}\}$$

- $G^{\mathbb{F}}_{n,k}$ is a compact $(2)nk$ smooth manifold.

Definition (tautological bundle $\gamma_{n,k}$)

$$E_{\gamma_{n,k}} = \left\{(K, v) \in G^{\mathbb{F}}_{n,k} \times \mathbb{F}^{n+k} | \ v \in K\right\}$$

$$p : E_{\gamma_{n,k}} \twoheadrightarrow G^{\mathbb{F}}_{n,k}$$

$$p(K, v) = K$$

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Unoriented Bordism

Definition unoriented bordism

Let $X \in Ob(\mathbf{CW})$. Let M, N be any compact smooth n-manifold. Let $f : M^+ \to X$, $g : N^+ \to X$. (M, f) is **cobordant** to (N, g) if $\exists W$ a compact smooth $n+1$-manifold wit boundary with $F : W^+ \to X$ such that:

- $\partial W = M \sqcup N$
- $F|_M = f$, $F|_N = g$

Definition unoriented bordism homology

$$\Omega_n^O(-) : \mathbf{HoCW} \to \mathbf{Ab}$$

$$\Omega_n^O(X) = \Big\{(M, f)|\ M \text{ compact smooth } n\text{-manifold },\ f : M^+ \to X\Big\}/_{\sim_{Cob}}$$

$$[M, f] + [N, g] = [M \sqcup N, f \sqcup g].$$

$$\Omega_n^O(f)[M, g] = [M, f \circ g]$$

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Stably complex manifolds

Definition stably complex manifolds

Let M be a smooth k manifold. We say that M is **stably complex** if for some $n \in \mathbb{N}$, there exists an isomorphism such that

$$\mathbf{N}(M, \mathbb{R}^{2n+k}) \cong \xi$$

with ξ a n complex vector bundle. We usually note this (M, ξ).

- Every complex manifolds of dimension n, seen as $2n$ real manifold, are stably complex.
- If $\mathbf{N}(M, \mathbb{R}^{2n+k})$ is complex, then $\mathbf{N}(M, \mathbb{R}^{2(n+1)+k}) = \mathbf{N}(M, \mathbb{R}^{2n+k}) \oplus \epsilon_{\mathbb{R}}^2 \cong \xi \oplus \epsilon_{\mathbb{C}}$.

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Unitary Cobordism

Definition (unitary bordism)

Let $X \in Ob(\mathbf{CW})$. Let $(M, \xi_M), (N, \xi_N)$ be any compact stably complex n-manifold. Let $f : M^+ \to X$, $g : N^+ \to X$. (M, ξ_M, f) is **unitary cobordant** to (N, ξ_N, g) if $\exists (W, \xi_W)$ a compact stably complex $n+1$-manifold with boundary and $F : W^+ \to X$ such that:

- $\partial W = M \sqcup N$
- $F|_M = f$, $F|_N = g$
- $\mathbf{N}(M, \mathbb{R}^{2w+n+1}) \cong \iota_M^*(\xi_W) \oplus \pm\epsilon_{\mathbb{R}} \cong \xi_M \oplus \epsilon_{\mathbb{C}}^u \oplus \pm\epsilon_{\mathbb{R}}$ with $\epsilon_{\mathbb{R}}$ given by the induced orientation on ∂W.
- $\mathbf{N}(N, \mathbb{R}^{2w+n+1}) \cong \iota_N^*(\xi_W) \oplus \mp\epsilon_{\mathbb{R}} \cong \xi_N \oplus \epsilon_{\mathbb{C}}^v \oplus \mp\epsilon_{\mathbb{R}}$ with $\epsilon_{\mathbb{R}}$ also given by the induced orientation on ∂W.

Isomorphism in $3.$ and $4.$ are such that $\forall x \in M$ or N, $\varphi|_{p^{-1}(x)} \in GL_{2w+1}^+(\mathbb{R})$.

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Unitary cobordism group

Definition Unitary bordism group

Let $X \in Ob(\mathbf{CW})$. We define the n **unitary bordism group** on X as

$$\Omega_n^U(X) = \Big\{(M, \xi_M, f) | M \text{ compact stably complex } n\text{-manifold }, f : M^+ \to X\Big\}/_{\sim_{Cob}}$$

$$[M, \xi_M, f] + [N, \xi_N, g] = [M \sqcup N, \xi_M \sqcup \xi_N, f \sqcup g].$$

$$0 = [\emptyset] \text{ and } [M, \xi_M, f]^{-1} = [M, \overline{\xi_M}, f]$$

$$\forall b \in M, \exists U \ s.t. \ \iota_U^*(\overline{\xi_M}) \cong U \times \mathbb{C}^{n-1} \times \overline{\mathbb{C}}$$

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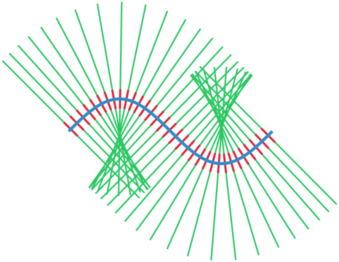
Thom-Pontrjagin construction

Tubular neighborhood theorem

Let W be a m dimensional smooth manifold, M a n dimensional embedded compact submanifold. Then, $\exists T$ open neighbourhood of M such that

$$T \cong \mathbf{N}(M, W)$$

with M is the zero section of this diffeomorphism.



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Thom-Pontrjagin construction

Thom-Pontrjagin construction

Let M be a compact stable complex manifold embedded into $\mathbb{R}^{2n+k}$. We have tubular neighborhood T with $\varphi : T \cong N(M, \mathbb{R}^{2n+k}) \cong \xi$. Then, seeing ξ as $int\big(D(\xi)\big)$, we get using Thom space,

$$\overline{\varphi} : S^{2n+k} \to T(\xi)$$

Using Thomification and universal representation theorem, we get what is called the **Thom-Pontrjagin construction**:

$$\Phi_M : S^{2n+k} \xrightarrow{\overline{\varphi}} T(\xi) \xrightarrow{T(j)} MU(n)$$

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Useful properties of Thom-Pontrjagin construction

1. This map is unique up to homotopy for a given M.

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2. If consider $\xi \cong \mathbf{N}(M, \mathbb{R}^{2n+k})$, then the Thom-Pontrjagin construction given by $\mathbf{N}(M, \mathbb{R}^{2(n+1)+k})$ is simply $\Sigma^2\Phi_M$.

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3. If $(M, \xi_M, f) \sim_{Cob} (N, \xi_N, g)$, then $\Phi_M \sim_{Hom} \Phi_N$.

Thom-Pontrjagin morphism

$$\Omega^U_{n,k}(X) = \{(M,f)|\ M \subset \mathbb{R}^{2n+k}, f : M^+ \to X\}/_{\sim_{Cob}}$$

with $(M,f) \sim_{Cob} (N,g)$ if $\exists W$ a cobordism with $W \subset \mathbb{R}^{2n+k} \times [0,1]$

$$\Omega^U_n(X) = \text{colim}_k\ \Omega^U_{n,k}(X)$$

Thom-Pontrjagin morphism

$$\Phi : \Omega^U_{n,k}(X) \to \pi_{2n+k}(X^+ \wedge MU(n))$$

$$[M,f] \to [\Phi_M]$$

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Thom-Pontrjagin isomorphism

Thom-Pontrjagin isomorphism

$$\Phi : \Omega^U_{n,k}(*) \cong \pi_{2n+k}(MU(n))$$

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Transversality

Let $f : M \to N$, $g : V \to N$ be any smooth maps. We say that f **is transversal to** g if whenever $f(p) = g(q)$

$$Df(T_pM) + Dg(T_qV) = T_{f(p)}N$$

with Df the smooth pushforward. We note transversality as $f \pitchfork g$.

If $f \pitchfork g$, $f^{-1}\big(g(V)\big)$ is a regular submanifold of M.

Thom transversality theorem

Let $f : M \to N, g : V \to N$ be two smooth maps.

$$\exists \widetilde{f} : M \to N, \widetilde{f} \sim_{Hom} f,\ \widetilde{f} \pitchfork g$$

Useful Observations

1. $$MU(n) = \text{colim}_k T(\gamma_{n,k})$$
2. $E_{\gamma_{n,k}}$ is a $k(n+1)$ complex manifold with $G^{\mathbb{C}}_{n,k}$ embedded in it.
3. $\mathbf{N}(G^{\mathbb{C}}_{n,k}, E_{\gamma_{n,k}}) \cong E_{\gamma_{n,k}}$.
4. $T(\gamma_{n,k}) \cong E^{\dagger}_{\gamma_{n,k}}$

Surjectivity

- $f : S^{2n+k} \to MU(n)$ is in fact $f : S^{2n+k} \to T(\gamma_{n,k}) \cong E^{\dagger}_{\gamma_{n,k}}$.

$$f|_{f^{-1}(E_{\gamma_{n,k}})} : U \to E_{\gamma_{n,k}}.$$

with $U \subset \mathbb{R}^{2n+k}$

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- Thus, by smooth approximation theorem $\exists \overline{f} : U \to E_{\gamma_{n,k}}$ with $f \sim_{Hom} \overline{f}$ and $\overline{f}$ smooth.
- Using Thom transversality theorem, $\exists \tilde{f} : U \to E_{\gamma_{n,k}}$ s.t.

$$\tilde{f} \pitchfork G^{\mathbb{C}}_{n,k}$$

$$\tilde{f} \sim_{Hom} f$$

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Surjectivity

$M = \tilde{f}^{-1}(G^{\mathbb{C}}_{n,k}) \subset U$ is a n compact manifold.

- M is stably complex.

$$\mathbf{N}(M, \mathbb{R}^{2n+k}) \cong \mathbf{N}(M, U) \cong \tilde{f}^*\Big(\mathbf{N}(G^{\mathbb{C}}_{n,j}, E_{\gamma_{n,j}})\Big) \cong \tilde{f}^*(\gamma_{n,j}).$$

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-

$$S^{2n+k} \xrightarrow{\ \overline{\varphi}\ } T\big(\mathbf{N}(M, \mathbb{R}^{2n+k})\big) \xrightarrow{\ T(i \circ \widetilde{f})\ } MU(n), \qquad S^{2n+k} \xrightarrow{\ f\ } MU(n)$$

$$\Phi_M \sim_{Hom} f$$

Injectivity

Let $H : S^{2n+k} \wedge [0,1]^+ \to MU(n)$ be an homotopy between Φ_M and Φ_N.

- $$\tilde{H} : U \times [0,1] \to E_{\gamma_{n,j}}$$
  $$\tilde{H} \pitchfork G^{\mathbb{C}}_{n,j}$$

Injectivity

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- $$\tilde{H} : U \times [0,1] \to E_{\gamma_{n,j}}$$
  $$\tilde{H} \pitchfork G^{\mathbb{C}}_{n,j}$$
- $W = \tilde{H}^{-1}(G^{\mathbb{C}}_{n,k})$ gives us a cobordism between M and N with
  $$\mathbf{N}(W, \mathbb{R}^{2n+k+1}) \cong \widetilde{H}^* \mathbf{N}(G^{\mathbb{C}}_{n,j}, E_{\gamma_{n,j}}) = \widetilde{H}^*(\gamma_{n,j}).$$

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