Master thesis

MU
Study of a Fundamental Spectrum in Homotopy Theory

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Introduction

Abstract:
This master thesis is dedicated to the study of the MU spectrum. To do so, it will first define spectra in homotopy theory and proving Brown's representation theorem. Then, it will study vector bundles, more precisely universal ones and their consequences on complex bordism. Finally, it will have a look at multiplicative structure on spectrum and the induced computation.

Quand on sait où l'on va, on va rarement très loin.[1]

This quote is a rather appropriate depiction of this master thesis. Indeed, despite its relatively intimidating length (at least compared to similar paper on the topic), this thesis covers somewhat "basic" questions and notions in its field, something apparent from the fact that most of the important theorems dates back from the 50's into the late 80's. But what this master project lacks in scope, it makes up in details and rigour as it is as complete as possible while in the framework of this master project.

To do so, this master thesis is decomposed into 4 chapters and a small appendix:

1. In **General Reminders**, we define properly basic notions that will be used throughout this paper, from pointed topological space and homotopy to CW complexes and (co)homology. For every interesting and important theorems, we quote the proofs in some references. They are fantastic reads to further our understanding on subjects that we have overlooked in this chapter as it is not the core of our paper.

2. The goal of **Spectra** is solely to define the central framework of our work, i.e. spectra (in the algebraic topology meaning of the word spectra). From its simplest definition, we will construct its category, its homotopy and shed light on many of its underlying structures. Then, we will investigate its first application by studying Brown's representation theorem.

3. In **Vector Bundles and** MU, we take somewhat of a break from spectra to study vector bundles, complex ones to be exact. Doing this, we define the Thom space and therefore finally define MU. To understand its importance, we will have a look at the notion

[1] René Thom

of universal bundle. Then, we will interest ourselves to a very interesting link between manifolds and MU using Thom-Pontrjagin construction.

4. **Multiplicative Structures** is dedicated to the study of multiplicative structures on cohomology and on spectra. Using those structures, it also studies the notion of orientation on a ring spectrum. Then, to use all of our previous theoretical constructions, we will compute cohomology of MU.

5. Finally, in the appendix **To go further in the study of MU**, we give further interesting properties of MU that we had not the time to further develop.

We will taking as a given that the reader is familiar with the following mathematical fields:

- Category theory, as defined in [ML13] first chapters.
- General topology, as defined in [Bre13] first chapter.
- Manifolds, as defined in [tD08] chapter 15 and [Kos13] chapter 1-4.
- Algebraic Topology, especially singular (co)homology as in [tD08] chapter 9 and 17.

Have a nice and pleasant reading.

Acknowledgement

I would like to give the biggest thanks possible [2] to the following people:

- To Jens Jakob Kjaer, whose unending support, quick thinking and illuminating indications made this project what it is today. He was always there for me, despite literal world end.
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[2] By assuming that this form a inductive poset and applying Zorn lemma

Contents

Chapter 1

General Reminders

Before going into the meat of this paper, we need to remind ourselves of many structures and results that will be building blocks of all our further work. We especially give here reminders on homotopy, (co)homology and CW complexes. The laters, being introduced during the 50s, are now mainstays of algebraic topology. We will explain why when working on them.

1.1 Pointed Spaces

In this paper, we will mainly work on the category of pointed topological space.

Definition 1.1.1 (Pointed Spaces).
We define the category of ***pointed topological spaces*** *noted* $\mathbf{Top}_\bullet$

$$Ob(\mathbf{Top}_\bullet) = \Big\{(X, x_0)|\ X \text{ is a topological space and } x_0 \in X\Big\}$$

$$\mathbf{Top}_\bullet\big((X, y_0), (Y, y_0)\big) = \Big\{f : X \to Y|\ f \text{ continuous and } f(x_0) = y_0\Big\}$$

This category is very similar to the one of topological space, **Top**. In fact, there is an adjunction between the two category using the following functor.

Definition 1.1.2 ($(-)^+$ functor).

$$(-)^+ : \mathbf{Top} \to \mathbf{Top}_\bullet$$

$$X^+ = (X \sqcup *, *)$$

$$f^+ = f \sqcup *$$

The $(-)^+$ functor is left adjoint with the forgetful functor. That is to say

$$\mathbf{Top}_\bullet(X^+, (Y, y_0)) \cong \mathbf{Top}(X, Y)$$

From now on, we will drop writing the base point for pointed spaces when it is clear from context.
Likewise to $\sqcup$ and $\times$ in **Top** or $\oplus$ and $\otimes$ in $M(R)$ (the module category on a ring), we define on $\mathbf{Top}_\bullet$ two types of product.

Definition 1.1.3 (Smash Product).
*Let $(X, x_0), (Y, y_0) \in Ob(\mathbf{Top}_\bullet)$, we define the **smash product** of X and Y*

$$X \wedge Y = X \times Y/_\sim$$

with $(x, y_0) \sim (x_0, y)$.
We give $X \wedge Y$ the quotient topology. $(X \wedge Y, [x_0, y_0]) \in Ob(\mathbf{Top}_\bullet)$.

Definition 1.1.4 (Wedge Product).
*Let $\{(X^\alpha, x_0^\alpha)\}_\alpha \in Ob(\mathbf{Top}_\bullet)$, we define the **wedge product** of $\{X^\alpha\}_\alpha$ as*

$$\bigvee_\alpha X^\alpha = \{\mathbf{x} \in \prod_\alpha X^\alpha, \mathbf{x}^\alpha = x_0^\alpha \text{ all but one time}\}$$

We give $\bigvee_\alpha X^\alpha$ the subset topology of $\prod_\alpha X^\alpha$. $(\bigvee_\alpha X^\alpha, (x_0^\alpha)_\alpha) \in Ob(\mathbf{Top}_\bullet)$

Notation 1.1.5.
We see that the equivalence class of $[x_0, y_0] \in X \wedge Y$ is just $X \vee Y$. We thus can write the smash product as the quotient set

$$X \wedge Y = X \times Y/X \vee Y$$

Proposition 1.1.6.
Let $X, Y, Z \in Ob(\mathbf{Top}_\bullet)$ be locally compact Hausdorff spaces, then smash product is associative

$$X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$$

$$(\bigvee_\alpha X^\alpha) \wedge Y = \bigvee_\alpha (X^\alpha \wedge Y)$$

Furthermore, we see that $\wedge$ *and* $\vee$ *are extension of* $\sqcup$ *and* $\times$ *in* $\mathbf{Top}_\bullet$*, meaning*

$$X^+ \wedge Y^+ \cong (X \times Y)^+$$

$$X^+ \vee Y^+ \cong (X \sqcup Y)^+$$

Now, we consider cones and suspension. Those are a central notion in homotopy theory.

Definition 1.1.7 (Cones)**.**
Let $(X, x_0) \in Ob(\mathbf{Top}_\bullet)$*,* $I = \big([0,1], 0\big)$ *we define the* ***cone*** *of* X

$$CX = X \wedge I$$

Now, let $f \in \mathbf{Top}_\bullet\Big((X, x_0), (Y, y_0)\Big)$*, we define the* ***mapping cone*** *of* f

$$Y \cup_f CX = (CX \sqcup Y/_\sim, \{*\})$$

with $(x, 1) \sim f(x)$

Notation 1.1.8.
If $f = \iota$ *the standard inclusion from* A *into* X *with same base point, we write* $X \cup_f CA$ *as* $X \cup CA$

Definition 1.1.9 (Cofibre sequence)**.**
We name any sequence

$$X \xrightarrow{f} Y \xrightarrow{\iota} Y \cup_f CX$$

a ***cofibre sequence****.*

Definition 1.1.10 (Suspension).
Let $(X, x_0), (Y, y_0) \in Ob(\mathbf{Top\bullet})$, $f \in \mathbf{Top\bullet}(X, Y)$. *We define the suspension space of* X *as*

$$\Sigma X = X \times I^+/_\sim$$

with $(x_0, a) \sim (x_0, b)$, $(x, 0) \sim (y, 0)$ *and* $(x, 1) \sim (y, 1)$ *with base point given by* $[x_0, a]$.
This in fact gives us the suspension functor

$$\Sigma : \mathbf{Top\bullet} \to \mathbf{Top\bullet}$$

$$\Sigma(X, x_0) = \Sigma X$$

$$\Sigma f(x, t) = (f(x), t)$$

Proposition 1.1.11.

$$\Sigma X \cong CX \cup_X CX$$

$$\Sigma S^n \cong S^{n+1}$$

$$\Sigma X \cong X \wedge S^1$$

$$\Sigma^n X \cong X \wedge S^n$$

1.2 Homotopy

Now that we have define $\mathbf{Top}_\bullet$, we have all tools needed to define homotopy on $\mathbf{Top}_\bullet$. Thanks to suspension Σ and smash product $- \wedge -$, it has more properties than the usual homotopy on **Top**, making it more interesting to study.

Definition 1.2.1 (Homotopy).
Let $f, g \in \mathbf{Top}_\bullet\big(X, Y\big)$. *We say that* f ***is homotopic to*** g $(f \sim_{Hom} g)$ *if and only if*

$$\exists H \in \mathbf{Top}_\bullet(X \wedge I^+, Y) \text{ such that } H(x, 0) = f, H(x, 1) = g$$

Definition 1.2.2 (Homotopy equivalence).
Let $X, Y \in Ob(\mathbf{Top_\bullet})$. We say that X is ***homotopically equivalent*** *to Y if $\exists f \in \mathbf{Top_\bullet}(X, Y), g \in \mathbf{Top_\bullet}(Y, X)$ such that*

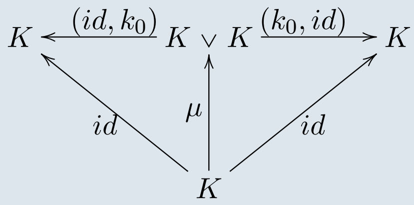
$$g \circ f \sim_{Hom} id_X, f \circ g \sim_{Hom} id_Y$$

Proposition 1.2.3.
Homotopy and homotopy equivalence are equivalence relations.

Now that we have a definition of homotopy, we want to give it more structure. A way to do it is by using H-cogroup.

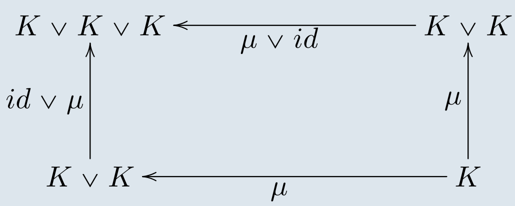
Definition 1.2.4 (H-cogroup).
We define an ***H-cogroup*** *as a $K \in Ob(\mathbf{Top_\bullet})$ equip with co-multiplication $\mu : K \to K \vee K$ and inverse map $\nu : K \to K$ such that:*

- *Consider $(id, k_0) : K \vee K \to K$, $(id, k_0)(k, k_0) = k, (id, k_0)(k_0, k) = k_0$. We muss have that the following diagram:*

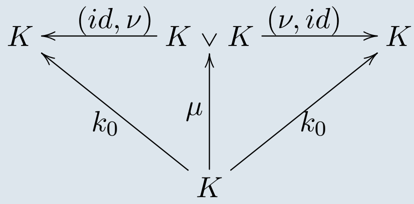
  is commutative up to homotopy.

- *The following diagram*

  is commutative up to homotopy.

- *The following diagram*



  is commutative up to homotopy.

- *We say that K is an H-commutative cogroup if μ and $\mu \circ T$ are homotopic. $T : K \vee K \cong K \vee K$. $T(x, y) = (y, x)$.*

Note that there also exists a notion of cogroup that is the same expect diagram commute strictly. Similarly, there exists a notion of H-group, where the multiplication is associative up to homotopy.
Now, we consider the central example of H-cogroup.

Example 1.2.5.
$\forall X \in Ob(\mathbf{Top_\bullet})$, $(\Sigma X, \mu', \nu')$ form an H-cogroup. with

$$\mu' : \Sigma X \to \Sigma X \vee \Sigma X$$

$$\mu'([x,t]) = \begin{cases} ([x, 2t], x_0) & 0 \leqslant t \leqslant \frac{1}{2} \\ (x_0, [x, 2t-1]) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

$$\nu' : \Sigma X \to \Sigma X$$

$$\nu'([x,t]) = [x, 1-t]$$

Furthermore, if $X = S^k$ with $k \geqslant 1$. Then $(\Sigma X, \mu', \nu')$ form a commutative H-cogroup.

Notation 1.2.6.

$$[X, Y] = \mathbf{Top_\bullet}(X, Y)/_{\sim Hom}$$

$$\pi_n(X) = [S^n, X], n \in \mathbb{N}$$

Using previous example and the fact that $S^{n+1} = \Sigma S^n$, we have that $\pi_n(X)$ can be given the structure of a group (for $n \geqslant 1$) using multiplication given by the following sequence

$$\Sigma S^{n-1} \xrightarrow{\mu} \Sigma S^{n-1} \vee \Sigma S^{n-1} \xrightarrow{f \vee g} X \vee X \xrightarrow{\Delta} X$$

Now, we have to see that function on **Top$_\bullet$** induce transformation on homotopy sets.

Definition 1.2.7 (Pullback & pushforward)**.**
Let $X, Y, Z \in Ob(\mathbf{Top_\bullet})$, $f \in \mathbf{Top_\bullet}(X, Y)$, *we define the* ***pushforward*** *of* f

$$f_* : [Z, X] \to [Z, Y]$$

$$f_*\big([h]\big) = [f \circ h]$$

and the ***pullback*** *of* f

$$f^* : [Y, Z] \to [X, Z]$$

$$f^*\big([h]\big) = [h \circ f]$$

See that $(f \circ g)_* = f_* \circ g_*$ *and* $(f \circ g)^* = g^* \circ f^*$

We now define a central notion in homotopy theory.

Definition 1.2.8 (Weak homotopy equivalences)**.**
Let $f \in \mathbf{Top_\bullet}(X, Y)$*. We say that* f *is a* ***weak homotopy equivalence*** *if* $\forall n \in \mathbb{N}$

$$f_* : \pi_n(X) \cong \pi_n(Y)$$

X *and* Y *are said to be* ***weakly homotopically equivalent*** *if there exists such a* $f \in \mathbf{Top_\bullet}(X, Y)$ *or in* $\mathbf{Top_\bullet}(Y, X)$*.*

Note that thanks to how we defined our group law, we get that f_* is a group isomorphism.

Proposition 1.2.9.
Weak homotopy equivalence is not an equivalence relation. But, if $f : X \to Y$ *is a weak homotopical map and* $f \sim_{Hom} g$*, then so is* g*.*

The proof of 1.2.9 is analogous to the one for spectra 2.1.21. It is thus omitted.

We now define the Ω functor.

Definition 1.2.10 (Ω functor).
Let $X \in Ob(\mathbf{Top}_\bullet)$, we define the following pointed topological set

$$\Omega X = Hom_{\mathbf{Top}_\bullet}(S^1, X)^1$$

Because $\forall Y \in Ob(\mathbf{Top}_\bullet)$, $Hom(Y, -)$ is a functor, then so is Ω. $\forall f \in \mathbf{Top}_\bullet(X, Y)$

$$\Omega : \mathbf{Top}_\bullet \to \mathbf{Top}_\bullet$$

$$\Omega(X) = \Omega X$$

$$f_* = \Omega f : \Omega X \to \Omega Y$$

Furthermore, using the exponential law ([Swi17], 0.13), we get that

$$\Omega^n X = Hom(S^n, X).$$

. Note that Ω has an equivalent in **Top** written as LX, but with fewer good properties. (For example, it isn't an H-group).
Now, we define the homotopy categories on $\mathbf{Top}_\bullet$ as follows.

Definition 1.2.11 (Homotopy Categories).
We define both naive homotopy category $\mathbf{HTop}_\bullet$ and homotopy category of pointed spaces $\mathbf{HoTop}_\bullet$

$$Ob(\mathbf{HTop}_\bullet) = Ob(\mathbf{Top}_\bullet)$$

$$\mathbf{HTop}_\bullet(X, Y) = [X, Y]$$

$$Ob(\mathbf{HoTop}_\bullet) = Ob(\mathbf{Top}_\bullet)$$

$$\mathbf{HoTop}_\bullet(X, Y) = [X, Y][\mathcal{W}^{-1}]$$

where $[X, Y][\mathcal{W}^{-1}]$ is the set $[X, Y]$ localised on the class $\mathcal{W}$ of all weak equivalences. This means that we add an abstract inverse f^{-1} for every weak equivalence map $f : Y \to X$.[2]

In the category $\mathbf{HTop}_\bullet$, isomorphisms are equivalence maps and in $\mathbf{HoTop}_\bullet$, isomorphisms are generated by weak equivalences and their abstract inverses.

[1] also written as $\mathbf{Top}_\bullet(S^1, X)$
[2] Further details about Homotopy models can be found in [DS95].

Remark 1.2.12.
Σ and Ω naturally extend both into functors from $\mathbf{HTop}_\bullet$ to $\mathbf{HTop}_\bullet$ and into functors from $\mathbf{HoTop}_\bullet$ to $\mathbf{HoTop}_\bullet$

1.3 Homology & Cohomology

We can now define homology and cohomology on $\mathbf{Top}_\bullet$. To be exact, we define generalised reduced (co)homology. Historically speaking, homology was create as a way to ease the computation of homotopy, but has then developed itself into a new and vast field.

Definition 1.3.1 (Reduced Homology Theory).
A family $\{H_n : \mathbf{HTop}_\bullet \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ *of functors and natural equivalences* $\{\sigma_n : H_n \to H_{n+1} \circ \Sigma\}_{n\in\mathbb{Z}}$ *is called a reduced homology theory* H_* *if:*

- $\forall (A, x_0) \subset (X, x_0)$, $i : A \hookrightarrow X$ *standard inclusion and* $j : X \twoheadrightarrow X \cup CA$, *then*

  $$H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X \cup CA)$$

  is exact.

Homology theory can also follows some further properties, named axioms.
Likewise to set theory, in homotopy theory, we want to understand the structure of our homology theory, knowing that it follows some axioms.

Definition 1.3.2 (Axioms of Reduced Homology Theory).

- ***Wedge axiom:*** $\forall \{(X_\alpha, x_\alpha) | \alpha \in A\}$ *and inclusion* $i_\alpha : X_\alpha \hookrightarrow \bigvee_{\beta\in A} X_\beta$, *then* $\forall n \in \mathbb{Z}$, *we get*

  $$\oplus_\alpha H_n(i_\alpha) : \oplus_\alpha H_n(X_\alpha) \cong H_n(\bigvee_{\beta\in A} X_\beta)$$

- ***Weak Homotopy Equivalence (WHE) axiom:*** *If* $f : X \to Y$ *is a weak homotopy equivalence, then* $\forall n \in \mathbb{Z}$, *we have that*

  $$H_n(f) : H_n(X, x_0) \cong H_n(Y, y_0).$$

Remark 1.3.3.
Using mapping cylinder (full details are analogous in 2.1.42), we can assure that every cofibre sequence (defined in 1.1.9) is homotopically equivalent to another one with simple inclusion. Thus, for any cofibre sequence,

$$H_n(A) \xrightarrow{H_n(f)} H_n(X) \xrightarrow{H_n(j)} H_n(X \cup_f CA)$$

is exact.

In this paper, we will mainly work on the dual of homology, named cohomology.

Definition 1.3.4 (Reduced cohomology theory)**.**
A family $\{H^n : \mathbf{HTop_\bullet} \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ of contravariant functors and natural equivalences $\{\sigma^n : H^{n+1} \circ \Sigma \to H^n\}_{n\in\mathbb{Z}}$ is called a reduced cohomology theory H^* if:

- *$\forall (A, x_0) \subset (X, x_0)$, $i : A \hookrightarrow X$ standard inclusion and $j : X \twoheadrightarrow X \cup CA$, then*

  $$H^n(A) \xleftarrow{H^n(i)} H^n(X) \xleftarrow{H^n(j)} H^n(X \cup CA)$$

  is exact.

Likewise to homology, it also as some axioms

Definition 1.3.5 (Axioms of Reduced Cohomology Theory)**.**

- ***Wedge axiom:*** *$\forall \{(X_\alpha, x_\alpha) | \alpha \in A\}$ and inclusion $i_\alpha : X_\alpha \hookrightarrow \bigvee_{\beta\in A} X_\beta$, then $\forall n \in \mathbb{Z}$, we get*

  $$H^n(i_\alpha) : H^n(\bigvee_{\beta\in A} X_\beta) \cong \oplus_\alpha H^n(X_\alpha)$$

- ***Weak Homotopy Equivalence (WHE) axiom:*** *If $f : X \to Y$ is a weak homotopy equivalence, then $\forall n \in \mathbb{Z}$, we have that*

  $$H^n(f) : H^n(Y, y_0) \cong H^n(X, x_0).$$

Remark 1.3.6.
We see that if (co)homology $H_*(H^*)$ satisfy the WHE axiom, then they can be fully defined as functors (contravariant functors)

$$H_* : \mathbf{HoTop_\bullet} \to \mathbf{Ab}$$

$$H^* : \mathbf{HoTop_\bullet} \to \mathbf{Ab}$$

Definition 1.3.7 (Unreduced (co)homology)**.**
*In this paper, we will work mainly on reduced (co)homology, but there exists a dual notion on unpointed topological space, named **unreduced (co)homology**.*
Let $\tilde{H}^*$ be a reduced cohomology, we define the dual unreduced cohomology H^* as

$$H^*(X) = \tilde{H}^*(X^+)$$

Let H^* be an unreduced cohomology, we define the dual reduced cohomology $\tilde{H}^*$ as given by

$$\tilde{H}^*(X) \oplus H^*(*) \cong H^*(X)$$

This is because of the cofibre sequence

$$S^0 \to X^+ \to X$$

All those definitions gives us many important and interesting results. One of them will be useful later in this paper [3], so we give it here.

Proposition 1.3.8.
Let H^* be a reduced cohomology [4] that follow the weak homotopy axiom, then, for any $A_1, A_2 \subset X$ with $A_1 \cup A_2 = X$ and $x_1 \in H^*(A_1), x_2 \in H^*(A_2)$ such that

$$H^*(i_{A_1 \cap A_2})(x_1) = H^*(i_{A_1 \cap A_2})(x_2).$$

Then, $\exists y \in H^*(X)$ such that $H^*(i_{A_1})(y) = x_1, H^*(i_{A_2})(y) = x_2$.

[3] More precisely in the construction of a representation of cohomology using Brown theorem 2.2.16.
[4] A similar result to proposition 1.3.8 also exists for reduced homology

We also define applications between (co)homology. Because (co)homotopy are functors, theses are natural transformations with a bit more structure.

Definition 1.3.9 (Natural Transformation of (co)homology).
A ***natural transformation*** $T_* : h_* \to k_*$ *between reduced homology theories (* $T^* : h^* \to k^*$ *between reduced cohomology theories) is a collection of natural transformations* $T_n : h_n \to k_n$ $(T^n : h^n \to k^n)$ *such that* $\forall X \in Ob(\mathbf{Top_\bullet})$*, the following diagram commute:*

$$
\begin{array}{ccc}
h_n(X) & \xRightarrow{\ \sigma\ } & h_{n+1}(\Sigma X) \\
{\scriptstyle T_n(X)}\downarrow & & \downarrow{\scriptstyle T_{n+1}(\Sigma X)} \\
k_n(X) & \xRightarrow{\ \sigma'\ } & k_{n+1}(\Sigma X)
\end{array}
\qquad
\begin{array}{ccc}
h^n(X) & \xLeftarrow[\ \sigma\ ]{} & h^{n+1}(\Sigma X) \\
\downarrow{\scriptstyle T^n(X)} & & \downarrow{\scriptstyle T^{n+1}(\Sigma X)} \\
k^n(X) & \xLeftarrow[\ \sigma'\ ]{} & k^{n+1}(\Sigma X)
\end{array}
$$

T_* (T^*) *is a* ***natural equivalence of reduced (co)homology theories*** *if each* T_n (T^n) *is a natural equivalence,* $n \in \mathbb{Z}$.

1.4 CW-Complexes

Now, we have a small problem. It occurs that many interesting properties in homotopy theory work only on weak homotopy theory. But **HoTop$_\bullet$** is a category not very practical to use. Thankfully, J. H. C. Whitehead has, during the 50s, define an object that is such a nice and powerful solution to this problem that it has become the basic ground of all following algebraic topological. Those are CW-complexes!

In all this part, let X be Hausdorff.

Definition 1.4.1 (Cell-complexes).
A ***cell-complex*** K *on* (X, x_0) *is a collection of subsets of* X *construct by induction on the* n***-skeleton*** K^n*:*

- $K^{-1} = \{x_0\}$
- $\forall n \in \mathbb{N}$*, we consider* $f_\alpha : S^{n-1} \to K^{n-1}$*. Then, we define* ***cells of dimention*** n e^n_α *as subspace of* X *that are equivalent to* D^n *glued to* K^{n-1} *along their boundary using* f_α*.*
  This means that their interior are homeomorphic and $f_\alpha : S^{n-1} \twoheadrightarrow \partial e^n_\alpha$

To be a cell-complex of X, K must also follows the following properties:

1. $X = \bigcup_{n \in \mathbb{N}, \alpha} e_\alpha^n$. *(it is usually noted $|K|$)*
2. $\dot{e}_\alpha^n \cap \dot{e}_\beta^m \neq \emptyset \Rightarrow n = m, \alpha = \beta$. *(with $\dot{e}_\alpha^n = int(e_\alpha^n)$)*

*A **subcomplex** $K' \subset K$ is a cell-complex such that $K'^n \subset K^n$. It may not necessary be the whole space X.*
We also define $K^{(n)} = K^n \backslash K^{n-1}$.
*Finally, we say a cell-complex is **finite** if it as finitely many cells.*

Definition 1.4.2 (CW-complexes)**.**
*A **CW-complex** is a cell complex K on X such that:*

- C) *K is **closure-finite**. i.e:*

$$e_\alpha^n \cap e_\beta^m = \emptyset \text{ except on finitely many occasions.}$$

- W) *X has the **weak topology** induced by K. i.e:*

$$S \subseteq X \text{ is closed } \iff \forall n \in \mathbb{N}, \alpha \in J_n, S \cap e_\alpha^n \text{ is closed in } e_\alpha^n$$

We usually don't distinguish between X and the complex K.

Proposition 1.4.3 (CW-subcomplexes)**.**
Let X be a CW-complex and Y a cell subcomplex, then Y is also a CW-complex with $|Y|$ closed in $|X|$

Remark 1.4.4.
CW complexes have a very interesting property. For any $k \in \mathbb{N}$, using skeleton, the sequence

$$X^k \to X^{k+1} \to \bigvee_\alpha S_\alpha^{n+1}$$

is a cofibre sequence.

As per usual, we also need to define maps between CW-complexes.

Definition 1.4.5 (Cellular Maps).
Let X, Y be CW-complexes, $f : X \to Y$ a continuous map is said ***cellular*** *if* $\forall n \in \mathbb{N}\ f(X^n) \subset Y^n$

Because CW complexes are pointed topological subset, we need to ask ourselves how our previous construction behave.

Proposition 1.4.6 (Properties of the smash/wedge product on CW-complexes).

- $\forall X, Y$ *CW-complexes,* $X \wedge Y$ *is also a CW-complex with*

$$(X \wedge Y)^{(n)} = \bigcup_{i+j=n} X^{(i)} \wedge Y^{(j)}$$

- *The smash product is associative on CW-complex.*
- *Let* $f : X \to Z, g : Y \to V$ *be cellular maps, then* $f \wedge g : X \wedge Y \to Z \wedge V$ *is also cellular.*
- $\forall\ X_\alpha$ *CW-complexes* $\bigvee_\alpha X_\alpha$ *is also a CW-complex with*

$$(\bigvee_\alpha X_\alpha)^{(n)} = \bigcup_\alpha X_\alpha^{(n)}$$

- *Let* $f_\alpha : X_\alpha \to Y_\alpha$ *be cellular maps, then* $\bigvee_\alpha f_\alpha : \bigvee_\alpha X_\alpha \to \bigvee_\alpha Y_\alpha$ *is also cellular.*

We see that thanks to this result, homotopy is well defined on CW complexes, seeing it as a restriction of homotopy on $\mathbf{Top}_\bullet$.
Now, a reasonable question is to consider the difference between continuous functions and cellular maps between CW-complexes. The following theorem answer our question.

Theorem 1.4.7 (Cellular Approximation Theorem).
Let X, Y be CW-complexes.

$$\forall f : X \to Y \text{ continuous, } \exists g : X \to Y \text{ cellular such that } f \sim_{Hom} g$$

A proof is given in [Hat02], page 349, Theorem 4.8.

This theorem gives us that we don't lose any information on homotopy of CW by looking only at cellular maps. Thus, we define their category.

Definition 1.4.8 (**CW** category)**.**
We define the category of CW complexes **CW**

$$Ob(\mathbf{CW}) = \Big\{X|\ X \text{ is a CW-complex}\Big\}$$

$$\mathbf{CW}(X,Y) = \big\{f : X \to Y \text{ cellular maps}\big\}$$

Proposition 1.4.9.
Σ *can also be seen as a functor from* **CW** *to* **CW**

$$\Sigma X = X \wedge S^1$$

$$\Sigma f = f \wedge id_{S^1}$$

We now ask ourselves what is the structure of $Hom(X,Y)$ with $X, Y \in Ob(\mathbf{CW})$?

Theorem 1.4.10 (Milnor's Theorem)**.**
Let X, Y be CW-complexes. Then, if X is finite, $Hom(X,Y)$ is homotopically equivalent to a CW complex.

This was proven in the following article [Mil59].

Corollary 1.4.11.
From theorem 1.4.10, we get that

$$\Omega : \mathbf{CW} \to \mathbf{CW}$$

is a well defined functor.

Now, another question we have is how big is **CW** compared **Top•** ?

Theorem 1.4.12 (CW-Approximation Theorem).

$$\forall (X, x_0) \in \mathbf{Top_\bullet}, \exists Y \in \mathbf{CW}, f : Y \to X \text{ a weak homotopy equivalence}$$

A proof can be found in [Hat02], page 352.

From the theorem 1.4.12 and proposition 1.4.6 , we get that we can define the homotopy on **CW** the same way as for **Top•**. We also define $[X, Y]$ and homotopy equivalence similarly to **Top•**.

Definition 1.4.13 (CW-Homotopy Category).
We define the CW-Homotopy Category **HoCW***:*

$$Ob(\mathbf{HoCW}) = Ob(\mathbf{CW})$$

$$\mathbf{HoCW}(X, Y) = [X, Y]$$

From this definition, we have that Σ and Ω are still well defined functors on **HoCW**. Furthermore, we have that **HoCW** inherit of some properties of **HTop•**. Namely

Lemma 1.4.14.
Let $X, Y, Z \in$ **CW** *such that Y is finite. Then, there is a natural equivalence*

$$A : [X \wedge Y, Z] \cong [X, Hom(Y, Z)]$$

$$A([f(x, y)]) = [\hat{f}(x)], \hat{f}(x)(y) = f(x, y)$$

Proof in [Swi17], 2.5 page 12.

Corollary 1.4.15.
From the previous lemma 1.4.14, we get that

$$A : [\Sigma X, Y] \cong [X, \Omega Y]$$

i.e. Σ and Ω are respectively left and right adjoint functors in **HoCW***.*

Lemma 1.4.16 (Homotopy extension property)**.**
Let $X, A, Y \in Ob(\mathbf{CW})$ such that $A \subset X$. Let $F : X \to Y$ and $g : A \to Y$ such that $F|_A \sim_{Hom} g$. Then, $\exists G : X \to Y$ such that $G|_A = g$ and $F \sim_{Hom} G$.

Some proof can be found in [Hat02], page 14.

Now, if we have seen that **CW** has a lot of very interesting properties, the following theorem is what makes it a very fundamental category in Homotopy Theory.

Theorem 1.4.17 (Whitehead theorem)**.**
Let $X, Y \in \mathbf{CW}, f \in \mathbf{Top}_\bullet(X, Y)$.

$$f : X \to Y \text{ is a weak homotopy equivalence} \iff f \text{ is an homotopy equivalence.}$$

[Hat02], page 346, Theorem 4.5, [Swi17], 6.32 page 89.

Corollary 1.4.18.

$$\mathbf{HoTop}_\bullet \cong \mathbf{HoCW}$$

By $\cong$, we mean that those two categories are equivalents, as described in [ML13], page 18.

This means that we have an equivalent category were all morphism are maps between sets.

Also, we see that we can thus write any reduced Homology and Cohomology that follows the weak homotopy equivalence axiom as functors and contravariant functors from $\mathbf{HoCW} \to \mathbf{Ab}$. This refinement gives us some very powerful properties:

Theorem 1.4.19.
Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of CW complexes that inject into one others. Let $X = colim_n X_n$. Then

$$\{(\iota_n)_*\} : \ colim_n \pi_k(X_n) \cong \pi_k(X)$$

The proof of this theorem can be found in [Swi17], 7.52.

Theorem 1.4.20.
Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of CW complexes that inject into one others, H_* be a reduced homology. Let $X = \ colim_n X_n$. Then

$$\{H_*(\iota_n)\}:\ colim_n H_*(X_n) \cong H_*(X)$$

The proof of this theorem can be found in [Swi17], 7.53.
Interestingly, this theorem is not as strait-forward when considering colimit and cohomology.

Theorem 1.4.21.
Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of CW complexes that inject into one others using $j_n : X_n \hookrightarrow X_{n+1}$. Let also H^* be a reduced cohomology that follows the wedge axiom. Let $X = \ colim_n X_n$. Then, we have the following short exact sequence:

$$0 \to \overset{1}{\lim} H^{q-1}(X_n) \to H^q(X) \xrightarrow{\{H^q(i_n)\}} \ lim_n H^q(X_n) \to 0$$

with $\lim^1 H^{q-1}(X_n) = coker(\delta)$

$$\delta : \prod_{n\in\mathbb{N}} H^q(X_n) \to \prod_{n\in\mathbb{N}} H^q(X_n)$$

$$\delta(f(n)) = (-1)^n f(n) + (-1)^{n+1} j_n^*(f(n+1))$$

with $f \in \prod_{n\in\mathbb{N}} H^q(X_n)$.

A proof is given in [Swi17], 7.66.

Corollary 1.4.22 (Mittag-Leffler criterion)**.**
Let $\{X_n\}$ be a sequence of CW complexes that inject into one others using $j_n : X_n \hookrightarrow X_{n+1}$. Let also H^* be a reduced cohomology that follows the wedge axiom. We note $j_n^m : X_n \hookrightarrow X_m$, $m > n$ the composition $j_{m-1} \circ \cdots \circ j_n$.
If for any $n \in \mathbb{N}$, $\exists N$ such that $\forall m \geqslant N$,

$$(j_n^m)^*\big(H^q(X_m)\big) = (j_n^N)^*\big(H^q(X_N)\big)$$

Then, $\lim_n^1 H^q(X_n) = 0$.

Remark 1.4.23.
Using previous definition of $\lim^1$, let $\{X_n\}_{n\in\mathbb{N}}, \{Y_n\}_{n\in\mathbb{N}}, \{Z_n\}_{n\in\mathbb{N}}$ be sequence of CW complexes that inject into one others and with maps

$$\phi_n : X_n \to Y_n$$

$$\psi_n : Z_n \to X_n$$

that agrees with each others. If $\forall n \in \mathbb{N}$, we have the s.e.s

$$0 \to H^q(Y_n) \xrightarrow{\phi_n^*} H^q(X_n) \xrightarrow{\psi_n^*} H^q(Z_n) \to 0$$

Then, using snake lemma, we get the exact sequence

$$0 \to \lim_n H^q(Y_n) \to \lim_n H^q(X_n) \to \lim_n H^q(Z_n) \to \lim_n^1 H^q(Y_n) \to \lim_n^1 H^q(X_n) \to \lim_n^1 H^q(Z_n) \to 0$$

Full details in [Swi17], 7.63.

Theorem 1.4.24 (Whitehead cohomology theorem)**.**
Let $T^* : k^* \to h^*$ be a natural transformation of cohomology[5] *theories satisfying the wedge axiom on* **HoCW**[6] *. If $T^q(S^0) : k^q(S^0) \to h^q(S^0)$ is an isomorphism for $q < n$ and an epimorphism for $q = n$, then, $\forall X \in Ob(\mathbf{CW})$*

$$T^q(X) : k^q(X) \to h^q(X)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$.

A proof is given in [Swi17], 7.67.

[5] A similar result exist also for natural transformation of homology. See [Swi17] 7.50.
[6] That is to say, it follows the weak homotopy equivalence axiom.

Chapter 2

Spectra

Now that we have reminded ourselves about homotopy, homology and CW-complexes, we can finally work on a definition of spectra. Historically speaking, they were introduced at the start of the 60s by Elon Lages Lima and refine by George W. Whitehead and J. Michael Boardman. They were initially defined as a way to give a proper category to computation like 2.1.23, but have since developed into their own field.

If in the first section, we will give a proper definition of spectra, we will also consider their immediate properties. Then, we will work during the second section on Brown's representation theorem, a result that link cohomology and spectra.

2.1 General Structure

This section will be divided in 4 subparts. In the first, we will build a solid definition of the spectra category. In the second one, we will show that spectra preserves many results of CW complexes. The third will be dedicated to group structure and in the final one we will show that spectra are an handy way to define (co)homology.

2.1.1 Definitions

Definition 2.1.1 (Spectra).
*A **spectrum** E is a collection $\{E_n\}_{n\in\mathbb{Z}}$ of CW complexes with injective singular maps*

$$p_n : \Sigma E_n \hookrightarrow E_{n+1}.$$

*A **subspectrum** $F \subset E$ is a subcollection $F_n \subset E_n$ such that $p_n(\Sigma F_n) \subset F_{n+1}$*

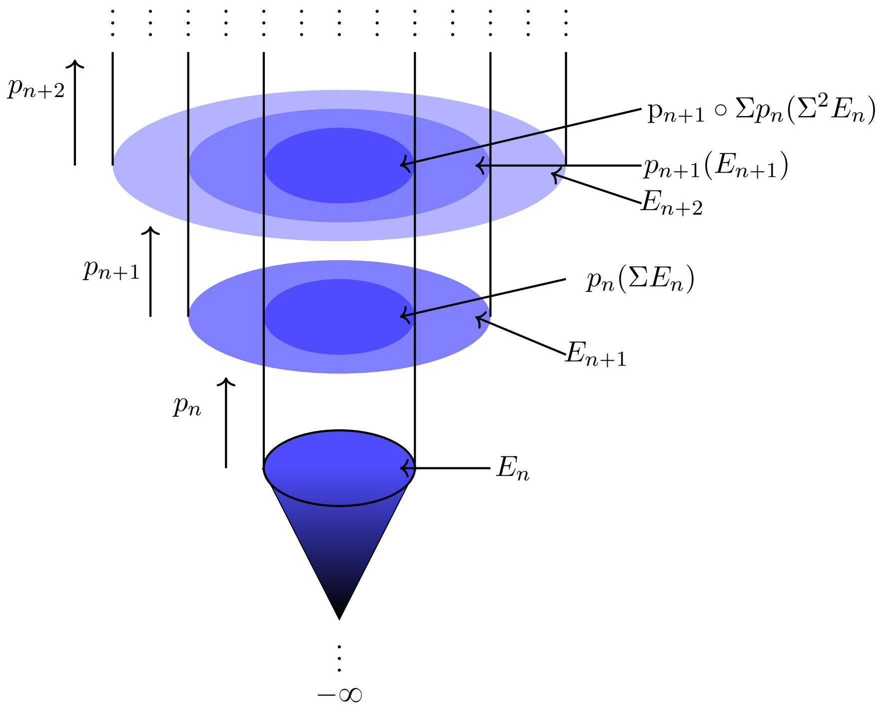


Figure 2.1: Representation of a Spectra

Example 2.1.2.

- *Let X be a CW-complex. We define $\Sigma^\infty(X)$ as:*

$$\Sigma^\infty(X) = \begin{cases} \{*\} & n < 0 \\ \Sigma^n X & n \geqslant 0 \end{cases}$$

- *We name this space if $X = S^0$ as $\Sigma^\infty(S^0) = \mathbb{S}$*

$$\mathbb{S} = \begin{cases} \{*\} & n < 0 \\ S^n & n \geqslant 0 \end{cases}$$

- *We define the spectrum U as $U_n = \bigvee_{i=0}^\infty S^i$ with $\Sigma U_n = \bigvee_{i=1}^\infty S^i \subset U_{n+1}$. It is interesting in the fact that it is rather counterintuitive, because it doesn't grow.*

Notation 2.1.3. *In order to shorten notations, we will write $p_n \circ \Sigma$ as Σ' when it is clear from context.*

Similarly to CW, spectra also have cells.

Definition 2.1.4 (Cells in spectra)**.**
Let E be a spectra and let e_n^d be a d-cell on E_n, then $\Sigma' e_n^d$ is a $d+1$-cell of E_{n+1} and so on. Then, going backward at most d time gives us $e_{n'}^{d'}$ (with $n = n' + d - d'$) with n' minimal. We thus define the cells on a spectrum as the following sequence:

$$e = (e_{n'}^{d'}, \Sigma' e_{n'}^{d'}, \Sigma'^2 e_{n'}^{d'}, \cdots)$$

Definition 2.1.5 (Size of a spectra)**.**
We define the size of a spectrum E as $size(E) = number\ of\ cells\ of\ E$.
*A spectrum E is called **finite** if it size is finite. It is **countable** if it size is countable.*

Also similarly to CW, spectrum have an analog to skeleton, called layers.

Definition 2.1.6 (Layers)**.**
*We define the **layers** E^n of a spectrum E by working with cells of E and using the following notion*

$$l(e) = \ min\{size(F) |\ F \text{ is a finite subspectrum of } E, e \in F\}$$

$$E^n = \bigcup_{l(e) \leqslant n} e$$

The following lemma lower somewhat the requirement needed to construct a spectrum. It is a very handy tool.

Lemma 2.1.7.
Let $\{E_n, p_n\}_{n\in\mathbb{Z}}$ be a collection of CW-complexes with cellular maps $p_n : \Sigma E_n \to E_{n+1}$. Then, we can construct a spectrum E' and homotopy equivalences $r_n : E'_n \to E_n$ such that the following diagram commute

$$\begin{array}{ccc} \Sigma E'_n & \xrightarrow{\iota} & E'_{n+1} \\ \downarrow{\scriptstyle \Sigma r_n} & & \downarrow{\scriptstyle r_{n+1}} \\ \Sigma E_n & \xrightarrow{p_n} & E_{n+1} \end{array}$$

Proof of Lemma 2.1.7:
Let E'_n be of the following form:

$$E'_n = E_n \wedge \{n\}^+ \cup \bigsqcup_{k<n} \Sigma^{n-k} E_k \wedge [k, k+1]^+_{/\sim}, n \in \mathbb{Z}$$

with $[x, k+1] \sim [\Sigma^{n-k-1} p_k(x), k+1]$. [1]
First, we see that

$$\Sigma E'_n = \Sigma E_n \wedge \{n\}^+ \cup \bigsqcup_{k<n} \Sigma^{n+1-k} E_k \wedge [k, k+1]^+_{/\sim} \subset \bigsqcup_{k<n+1} \Sigma^{n+1-k} E_k \wedge [k, k+1]^+_{/\sim} \subset E'_{n+1}$$

We define $r_n : E'_n \to E_n$ as :

$$r_n([x, t]) = p_{n-1} \circ \Sigma p_{n-2} \circ \cdots \circ \Sigma^{n-k-1} p_{n-k}(x) \text{ with } [x, t] \in \Sigma^{n-k} E_k \wedge [k, k+1]^+$$

It is well defined: indeed, if $t = k+1$, by the definition of r_n, it is the same. Furthermore, if $t = n$, then, $r_n([x, t]) = x$.
Furthermore, we define $i_n : E_n \to E'_n$

$$i_n(x) = [x, n] \in E_n \wedge \{n\}^+ \subset E'_n$$

Then, $r_n \circ i_n(x) = r_n([x, n]) = x$ so $r_n \circ i_n = id_{E_n}$. Now, let's show $i_n \circ r_n \sim_{Hom} id_{E'_n}$
We define $H : E'_n \wedge I^+ \to E'_n$

$$H([x, t], s) = \begin{cases} [\Sigma^{n-m} p_{m-1} \circ \Sigma^{n-m+1} p_{m-2} \circ \cdots \circ \Sigma^{n-k-1} p_k(x), (1-s)t + sn] \\ \text{with } [x, t] \in \Sigma^{n-k} E_k, k < n, m \leqslant (1-s)t + sn \leqslant m+1, k \leqslant m < n. \\ [x, n] \text{ for } [x, t] \in E_n \wedge \{n\}^+. \end{cases}$$

It is well defined similarly to r_n with $H \circ i_0 = id_{E'_n}$ and $H \circ i_1 = i_n \circ r_n$
Now, let's show that $r_{n+1}|_{\Sigma E'_n} = p_n \circ \Sigma r_n$. This comes directly

$$p_n \circ \Sigma r_n([x, t]) = {}^2 p_n \circ \Sigma p_{n-1} \circ \Sigma^2 p_{n-2} \circ \cdots \circ \Sigma^{n-k} p_{n-k}(x) = r_{n+1}|_{\Sigma E'_n}$$

Then, by considering $\{E'_n, \subset\} = E'$, we have our spectra.

□ 2.1.7

This lemma is usually used to define a sub-category of spectra (named CW-spectra) where $\Sigma E_n \subseteq E_{n+1}$. It can be proven using 2.1.37 that this subcategory is homotopically equivalent to the category of spectra. To somewhat streamline this paper, we won't study it in details.
Now, we will construct morphism between spectra. A first way to do it is as follows

Definition 2.1.8 (Functions on spectra)**.**
*Let E, F be spectra. A **function** $f : E \to F$ is a collection of cellular maps $\{f_n : E_n \to F_n\}_{n \in \mathbb{Z}}$ such that:*

$$f_{n+1}|_{\Sigma' E_n} = \Sigma' f_n$$

[1] with $[x, k+1] \in \Sigma^{n-k} E_k \wedge [k, k+1]^+$ and $[\Sigma^{n-k-1} p_k(x), k+1] \in \Sigma^{n-k-1} E_{k+1} \wedge [k+1, k+2]^+$.
[2] because Σ is a functor

Sadly, functions on spectra are too strict of a definition to have all the properties we desire. Indeed, if we see spectrum as sort of colimit, then we don't need to know where something is at every time, but just where it will be at some point.
To get this result, we need work a bit on what are called cofinal subspectra.

Definition 2.1.9 (Cofinal subspectra)**.**
*Let E be a spectrum, $F \subset E$ a subspectrum is called **cofinal** if $\forall e_n \in E_n, \exists m$ such that $\Sigma'^m e_n \in F_{n+m}$.*

Lemma 2.1.10.

- *Let F, G be two cofinal subspectra of E, then $F \cap G$ is also a cofinal subspectrum of E.*
- *Let G be a cofinal subspectrum of F, itself a cofinal subspectrum of E, then G is a cofinal subspectrum of E.*
- *Arbitrary unions of subspectra with at least one cofinal are cofinal subspectra.*

Proof of Lemma 2.1.10:

- First, we see that if F, G are subspectra, then so does $F \cap G$. Indeed
$$\Sigma'(F \cap G)_n = \Sigma' F_n \cap \Sigma' G_n \subseteq F_{n+1} \cap G_{n+1} = (F \cap G)_{n+1}$$
Now, $F \cap G$ is cofinal. Indeed, if $\Sigma'^m(e_n) \in F_{n+m}$, then $\forall k \in \mathbb{N}, \Sigma'^{m+k}(e_n) \in F_{n+m+k}$. Thus, $\forall e_n \in E_n$, using $m = \max(m_1, m_2)$ with m_i given by respective cofinality of F and G, we get
$$\Sigma'^m e_n \in F_{n+m} \cap G_{n+m} = (F \cap G)_{n+m}$$
- First, we see that G is a subspectra of E, indeed
$$\Sigma' G_n \subseteq F_{n+1} \subseteq E_{n+1}$$
Secondly, $\forall e_n \in E_n$, using $m = m_1 + m_2$ with m_1 given by cofinality of F in E and m_2 given by cofinality of F in E on the cell $(\Sigma'^{m_1} e_n)$, we get that:
$$\Sigma'^m e_n = \Sigma'^{m_2} \circ \Sigma'^{m_1} e_n \in G_{n+m}$$
- $\bigcup_{i \in I} F^i$ is a subspectra because:
$$\Sigma'(\bigcup_{i \in I} F^i)_n = \bigcup_{i \in I} \Sigma' F^i_n \subseteq \bigcup_{i \in I} F^i_{n+1} = (\bigcup_{i \in I} F^i)_{n+1}$$
Furthermore, it is cofinal using m given by the cofinal subspectra.

□ 2.1.10

We now can define what we call maps of spectra.

Definition 2.1.11 (Maps of spectra)**.**
Let E, F be spectra. We consider the following set S

$$S = \Big\{(E', f')|\ E' \text{ is a cofinal subspectra of } E, f' : E' \to F\Big\}$$

We define a equivalence relation on S:

$$(E', f') \sim (E'', f'') \Leftrightarrow \exists(\widetilde{E}, \widetilde{f}) \in S, \widetilde{E} \subset E' \cap E'' \text{is a cofinal subspectra and } \widetilde{f} = f'|_{\widetilde{E}} = f''|_{\widetilde{E}}$$

We define a ***map*** *from E into F as the equivalence class $[E', f']$ and we define*

$$Hom(E, F) = S/_{\sim}$$

Verification of Definition 2.1.11:
This is indeed an equivalence relation. We show the associativity. Consider $(E', f') \sim (E'', f'')$ using $(\widetilde{E}_1, f_1)$ and $(E'', f'') \sim (E''', f''')$ using $(\widetilde{E}_2, f_2)$.
Then consider $(\widetilde{E}_1 \cap \widetilde{E}_2, f''|_{\widetilde{E}_1 \cap \widetilde{E}_2}) \in S$.

$$\widetilde{E}_1 \cap \widetilde{E}_2 \subset E' \cap E'' \cap E''' \subset E' \cap E'''$$

$$f'|_{\widetilde{E}_1 \cap \widetilde{E}_2} = f_1|_{\widetilde{E}_2} = f''|_{\widetilde{E}_1 \cap \widetilde{E}_2} = f_2|_{\widetilde{E}_1} = f'''|_{\widetilde{E}_1 \cap \widetilde{E}_2}$$

So it give us that $(E', f') \sim (E', f'')$

□ 2.1.11

It is not apparent that maps can be composed with each other. To do so, we will need the following lemma.

Lemma 2.1.12.
Let E, F be spectra and $f : E \to F$ a function. If $F' \subset F$ is a cofinal subspectrum, then $\exists E' \subset E$ a cofinal subspectrum such that $f(E') \subset F'$.

Proof of Lemma 2.1.12:
Let S be the set of all subspectra $G \subset E$ such that $f(G) \subset F'$. Let $E' = \bigcup_{G \in S} G$. Then E' is a subspectrum of E with $f(E') \subset F'$. Let's show it is cofinal.
Consider $e_n \in E_n$. Let $e_n \in e$ a cell of E. Then consider V the finite subspectra of E such that $e \in V$[3]. Then, we have that $f(V)$ is therefore a finite subspectra of F.

[3] We can always find such a spectrum by induction using the closure finite and using skeleton

Then, there exists M such that $f(V_M) \subseteq F_M$. Then, $V^{\geqslant M} = \begin{cases} * & n < M \\ V_n & n \geqslant M \end{cases}$ is such that $f(V^{\geqslant M}) \subset F'$. Thus, $V^{\geqslant M} \subset E'$, proving the cofinality.

□ 2.1.12

Corollary 2.1.13.
We can composed maps of spectra.

Proof of Corollary 2.1.13:
Let E, F, G be spectra and let $[E', f] \in Hom(E, F), [F', g] \in Hom(F, G)$. Then, consider $f : E' \to F$. Using lemma 2.1.12 on F' , we get that $\exists E''$ a cofinal subspectra of E' and thus E such that $f(E'') \subset F'$ ($[E'', f|_{E''}] = [E', f]$). Therefore, we define the composition of spectral maps:

$$Hom(F, G) \times Hom(E, F) \to Hom(E, G)$$
$$[F', g] \circ [E', f] = [E'', g \circ f|_{E''}]$$

It is well defined. Indeed, take $(E', f) \sim (E'^*, f^*), (F', g) \sim (F'^*, g^*)$.
Then, using $(E'', f|_{E''}) \sim (E', f) \sim (E'^*, f^*) \sim (E''^*, f|_{E''^*})$, we get $(\widetilde{E}, \widetilde{f})$ $\widetilde{E} \subset E'' \cap E''^*$, $\widetilde{f} = f|_{\widetilde{E}} = f^*|_{\widetilde{E}}$ and $(\overline{F}, \overline{g})$, $\overline{F} \subset F' \cap F'^*$, $\overline{g} = g|_{\overline{F}} = g^*|_{\overline{F}}$. Thus, using lemma 2.1.12, we finally get $(\overline{E}, \overline{f})$ under $(\widetilde{E}, \widetilde{f})$.
Consider $(\overline{E}, \overline{g} \circ \overline{f})$.

$$\text{Then } \overline{E} \subset E'' \cap E''^* \text{and}$$
$$g \circ f|_{\overline{E}} = \overline{g} \circ f|_{\overline{E}} = \overline{g} \circ \widetilde{f}|_{\overline{E}} = \overline{g} \circ \overline{f}|_{\overline{E}} = \overline{g} \circ \widetilde{f}|_{\overline{E}} = \overline{g} \circ f^*|_{\overline{E}} = g^* \circ f^*|_{\overline{E}}$$

Thus,

$$[E'', g \circ f|_{E''}] = [E''^*, g^* \circ f^*|_{E''^*}]$$

□ 2.1.13

Now that we have our maps that commute, we can finally define the category of spectra.

Definition 2.1.14 (Spectra category)**.**
We define the category of spectra **Sp***.*

$$Ob(\mathbf{Sp}) = \{E |\ E \textit{ is a spectrum as defined in 2.1.1}\}$$

$$\mathbf{Sp}(E, F) = Hom(E, F)$$

with $id_E = [E, id_E]$

Remark 2.1.15.
Let $E \in Ob(\mathbf{Sp})$ and let $E' \subset E$ be a cofinal subspectrum. Consider $[E', id_{E'}] \in \mathbf{Sp}(E, E')$ and $[E', \iota] \in \mathbf{Sp}(E', E)$. Then

$$[E', id_{E'}] \circ [E', \iota] = [E', id_{E'}] = [E, id_E]$$

$$[E', \iota] \circ [E', id_{E'}] = [E', id_{E'}]$$

Thus, E and E' are isomorphic in $\mathbf{Sp}$.

Notation 2.1.16. *We usually write a map $f : E \to F$ without specifying it's cofinal domain because as showed in the previous remark, it is isomorphic to E in* $\mathbf{Sp}$*.*

Now, that we have our category $\mathbf{Sp}$, we can consider the following functors on it.

Definition 2.1.17 (Functors on $\mathbf{Sp}$)**.**

- *We define the functor Σ^∞:*

$$\Sigma^\infty : \mathbf{CW} \to \mathbf{Sp}$$

$$\Sigma^\infty(X) = \Sigma^\infty X$$

$$\Sigma^\infty(f) = \{f_n\}_{n\in\mathbb{Z}} \quad \textit{with } f_n = \begin{cases} id_{\{*\}} & n < 0 \\ \Sigma^n f : \Sigma^n X \to \Sigma^n Y & n \geqslant 0 \end{cases}$$

- *We define the functor Σ that shift our spectrum:*

$$\Sigma : \mathbf{Sp} \to \mathbf{Sp}$$

$$(\Sigma E)_n = E_{n+1}$$

$$(\Sigma f)_n = f_{n+1}$$

- *We see that this also induce an inverse functor Σ^{-1}*

$$\Sigma^{-1} : \mathbf{Sp} \to \mathbf{Sp}$$

$$(\Sigma^{-1} E)_n = E_{n-1}$$

$$(\Sigma^{-1} f)_n = f_{n-1}$$

2.1.2 Homotopy on spectra

Having our category **Sp**, we want, similarly to **CW**, to define an homotopy on it. For that, we need some preliminary definitions and results.

Example 2.1.18.
We give here a example of cofinal spectrum that is central in spectral homotopy theory.

All cofinal subspectra of $\Sigma^n \mathbb{S}$ *are of the form* $\Sigma^{n-r}\Sigma^\infty S^r$.

Definition 2.1.19 (Smash product of a spectrum with a CW complex)**.**
Let $X \in Ob(\boldsymbol{CW}), E \in Ob(\boldsymbol{Sp})$. *We define the smash product* $E \wedge X \in Ob(\boldsymbol{Sp})$*:*

$$(E \wedge X)_n = E_n \wedge X$$

$$\Sigma(E \wedge X)_n \cong S^1 \wedge (E_n \wedge X) \cong (S^1 \wedge E_n) \wedge X \cong \Sigma E_{n+1} \wedge X$$

$$\Sigma'(E \wedge X)_n \cong (p_n \wedge id_X)(\Sigma E_n \wedge X) \subset E_{n+1} \wedge X$$

Let $f : E \to F$ *be a map represented by* (E', f) *and* $g : X \to Y$ *a cellular map. Then, we define*

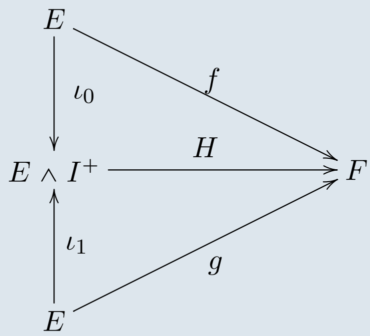
$$f \wedge g : E \wedge X \to F \wedge Y \text{ represented by } (E' \wedge X, f \wedge g)$$

Remark 2.1.20.
$\forall X \in Ob(\mathbf{CW})$

$$\Sigma^\infty X \cong \mathbb{S} \wedge X$$

We now can define spectral homotopy.

Definition 2.1.21 (Homotopy on spectra)**.**
Let $E, F \in Ob(\mathbf{Sp})$, $f, g \in \mathbf{Sp}(E,F)$. We say that f is ***homotopic*** *to g ($f \sim_{Hom} g$) if $\exists$ a map $H : E \wedge I^+ \to F$ such that the following diagram commute*



Meaning there exists a cofinal subspectra where the functions they're build out agree.
Homotopy equivalence is an equivalence relation. We also define the following sets

$$[E, F] = Hom(E, F)/_{\sim_{Hom}}$$

$$\pi_n(E) = [\Sigma^n \mathbb{S}, E] \; n \in \mathbb{Z}$$

Verification of Definition 2.1.21:
Homotopy is indeed an equivalence relation:

- *reflexivity*: Let $f : E \to F$ be defined as $[E', f']$. We define

$$H : E \wedge I^+ \to F$$

$$H = [E' \wedge I^+, f' \wedge id_X]$$

$$H \circ i_{0,1} = [\overline{E}, (f' \wedge id_X) \circ i_{0,1}] = [\overline{E}, f'] = [E', f'] = f$$

- *symmetry*: Let $H = [\overline{E}, \overline{H}]$ be an homotopy between $f = [E', f']$ and $g = [E'', g'']$. Then consider

$$\widetilde{H} = H \circ \big(id_E \wedge (1-t)\big) = [\widetilde{E}, \overline{H} \circ id_E \wedge (1-t)]$$

$$\widetilde{H} \circ i_n = [\widetilde{E}^*, \overline{H} \circ \big(id_E \wedge (1-t)\big) \circ i_n] = [\widetilde{E}^*, \overline{H} \circ i_{1-n}] = [\overline{E}^*, \overline{H} \circ i_{1-n}] = \begin{cases} [E'', g''] & n = 0 \\ [E', f'] & n = 1 \end{cases}$$

- *associativity*: Let $H_1 = [E_1, H_1]$ be the homotopy between f and g, $H_2 = [E_2, H_2]$ the homotopy between g and h. Then consider the following map

$$\overline{H} = [E_1 \cap E_2, \begin{cases} H_1 \circ (id_X \wedge 2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ H_2 \circ (id_X \wedge 1 - 2t) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}]$$

This gives us the homotopy between f and h.

□ 2.1.21

Similarly to **CW**, we can define pullbacks and pushforwards.

Definition 2.1.22 (Pullbacks & pushforwards on spectra)**.**
Let $E, F, G \in \mathbf{Sp}$, $f \in \mathbf{Sp}(E, F)$, *we define the* ***pushforward*** *of* f

$$f_* : [G, E] \to [G, F]$$

$$f_*\big([h]\big) = [f \circ h]$$

and the ***pullback*** *of* f

$$f^* : [F, G] \to [E, G]$$

$$f^*\big([h]\big) = [h \circ f]$$

See that $(f \circ g)_* = f_* \circ g_*$ *and* $(f \circ g)^* = g^* \circ f^*$

Now, we find the first important property of spectral homotopy. It is a central equivalence and a reason why spectra were defined in the first place.

Proposition 2.1.23.
Let $E \in Ob(\mathbf{Sp})$. *Using remark 2.1.18 and that a function* $f : \Sigma^{n-r}\Sigma^\infty S^r \to E$ *is just a map* $f : S^r \to E_{r-n}$ *with further parts defined by suspension* $\Sigma^s f : S^{s+r} \to \Sigma'^s E_{r-n} \subset E_{r-n+s}$. *Then*

$$\alpha : \pi_n(E) \cong colim_k \; \pi_{n+k}(E_k)$$
$$\alpha([\Sigma^{n-r}\Sigma^\infty S^r, f]) = \{f\} = \{\cdots, *, f, \Sigma f, \cdots\}$$

using for our colimit

$$\pi_{n+k}(E_k) \xrightarrow{\Sigma} \pi_{n+1+k}(\Sigma E_k) \xrightarrow{p_{n*}} \pi_{n+k+1}(E_{k+1})$$

Furthermore, under this form, we get that pushforward of a spectral function $r : E \to F$ *induce the following commutative diagram:*

$$\begin{array}{ccc} \pi_n(E) & \xrightarrow{\quad r_* \quad} & \pi_n(F) \\ \Big\Vert \; \alpha & & \alpha \; \Big\Vert \\ colim_k \pi_{n+k}(E_k) & \xrightarrow{\{r_{k*}\}} & colim_k \pi_{n+k}(F_k) \end{array}$$

Proof of Proposition 2.1.23:
This is well defined because $f : S^r \to E_{r-n}$ is unique.

- *surjective:* For any $\{[f_n]\} \in \text{colim}_k\ \pi_{n+k}(E_k)$, Then, $\exists [f] \in \pi_{n+r}(E_r)$ minimal such that $[f] \neq [*]$. Then, using f as a representative, we get $\alpha([\Sigma^{n-r}\Sigma^\infty S^r, f]) = \{f_n\}$.
- *injective:* Let $\{[f]\} = \{[g]\}$. Then, we can find an homotopy $H : S^{n+r} \wedge I^+ \to E_{r+s-n}$ between $\Sigma^s f$ and $\Sigma^s g$ for some s. Then, $[\Sigma^{n-r-s}S^{r+s} \wedge I^+, H]$ is an homotopy of $[\Sigma^{n-r}\Sigma^\infty S^r, f]$ and $[\Sigma^{n-r}\Sigma^\infty S^r, g]$.

For the last part, this comes from the fact that $\alpha \circ r_*(\mu) = \{[r_k \circ \mu]\} = \{r_{k*}\}\{[\mu]\} = \{r_{k*}\}\alpha(\mu)$.

□ 2.1.23

Because $\pi_{k+n}(E_n)$ are abelian groups, we can give $\pi_k(E)$ the structure of their colimit, making it an abelian group.
Now, we define, similarly to **CW**, the homotopy equivalence on sets.

Definition 2.1.24 (Homotopy equivalences on spectra)**.**

- *Let $E, F \in Ob(\mathbf{Sp})$, $f \in \mathbf{Sp}(E, F)$. We say that f is a **homotopy equivalence** between E and F if $\exists g \in \mathbf{Sp}(F, E)$ such that $f \circ g \sim_{Hom} id_F, g \circ f \sim_{Hom} id_E$.*

  *We say that E and F are **homotopically equivalent** ($E \sim_{Hom} F$) if there exists an homotopy equivalence $f \in [E, F]$.*
- *Let $E, F \in Ob(\mathbf{Sp})$, $f \in \mathbf{Sp}(E, F)$. We say that f is an **weak homotopy equivalence** between E and F if*
$$\forall n \in \mathbb{Z}, f_* : \pi_n(E) \cong \pi_n(F).$$

Verification of Definition 2.1.24:
Being homotopically equivalent is indeed an equivalence relation:

- *reflexivity*: $E \sim_{Hom} E$ using id_E.
- *symmetry*: $E \sim_{Hom} F$ using f and g. Then $F \sim_{Hom} E$ using g and f.
- *transivity*: $E \sim_{Hom} F$ using f and g and $F \sim_{Hom} H$ using f' and g'. Then consider $f' \circ f \in \mathbf{Sp}(E, H), g \circ g' \in \mathbf{Sp}(H, E)$. Then:
$$f' \circ f \circ g \circ g' \sim_{Hom} f' \circ id_F \circ g' \sim_{Hom} f' \circ g' \sim_{Hom} id_H$$
$$g \circ g' \circ f' \circ f \sim_{Hom} g \circ id_F \circ f \sim_{Hom} g \circ f \sim_{Hom} id_E$$

Furthermore, being an homotopy equivalence is stronger than being a weakly homotopy equivalence. Indeed, using $f : E \to F, g : F \to E$ given by $E \sim_{Hom} F$, we get that:

$$\forall [h] \in \pi_n(E), g_* \circ f_*[h] = (g \circ f)_*[h] = [h \circ g \circ f] = [h \circ id_E] = [h] \Rightarrow g_* \circ f_* = id_{\pi_n(E)}$$

and similarly for $f_* \circ g_*$. Hence, f and g are weak homotopy equivalence.

□ 2.1.24

Now, we define cones and wedge product on spectra.

Definition 2.1.25 (Mapping Cone on Spectra)**.**

- *Let E be a spectrum, we define the **cone** of E: $CE = E \wedge ([0,1], 0)$.*
- *Let $f : E \to F$ be a spectra map. We define the **mapping cone** of f noted $F \cup_f CE$:*

$$(F \cup_f CE)_n = F_n \cup_{f'_n} CE'_n \text{ with } (E', f') \text{ representing } f$$

We can see that for $X \in Ob(\mathbf{CW})$ $\Sigma^\infty CX \cong C\Sigma^\infty X$.

Remark 2.1.26.
Mapping cones on spectra are independent from the the choice of representation. Indeed, see that for $(E', f'), (E'', f'')$ both representing $f$with $(\overline{E}, \overline{f})$ under both, we get that $F \cup_{f'} CE'$ and $F \cup_{f''} CE''$ have for mutual cofinal subspectrum $F \cup_{\overline{f}} C\overline{E}$ and they are thus isomorphic in **Sp**.

Definition 2.1.27 (Wedge product on spectra)**.**
We define the wedge product on spectra. Let E^α be spectra

$$(\bigvee_{\alpha \in A} E^\alpha)_n = \bigvee_{\alpha \in A} E^\alpha_n$$

$$\Sigma' \Big(\bigvee_{\alpha \in A} E^\alpha_n \Big) = \bigvee_{\alpha \in A} \Sigma' E^\alpha_n$$

Proposition 2.1.28.
Let $i_\beta : E^\beta : \bigvee_{\alpha\in A} E^\alpha$ be the standard inclusion maps. Then, this induce the following bijections

$$\{- \circ i_\alpha\}_{\alpha\in A} : Hom(\bigvee_{\alpha\in A} E^\alpha, F) \cong \prod_{\alpha\in A} Hom(E^\alpha, F)$$

$$f \to \prod_{\alpha\in A}(f \circ i_\alpha)$$

$$\{i_\alpha^*\}_{\alpha\in A} : [\bigvee_{\alpha\in A} E^\alpha, F] \cong \prod_{\alpha\in A}[E^\alpha, F]$$

$$[f] \to \prod_{\alpha\in A}[f \circ i_\alpha]$$

Proof of Remark 2.1.28:
First, let $\overline{E} \subset \bigvee_{\alpha\in A} E^\alpha$ be a cofinal subspectrum. We have that $\overline{E} \cap E^\alpha$ is cofinal in E^α. Indeed, because cells $e \in E_n^\alpha$ are also cells of $(\bigvee_{\alpha\in A} E^\alpha)_n$ and because $\Sigma' e \in (\bigvee_{\alpha\in A} E^\alpha)_{n+1}$ is by definition in E_{n+1}^α, we get that $\exists m$ such that $\Sigma'^m e \in \overline{E} \cap E_{n+m}^\alpha$.
Inversely, with the same reasoning, let $\overline{E}^\alpha$ be a cofinal subspectrum of E^α. Then, we have that $\bigvee_{\alpha\in A} \overline{E}^\alpha$ is a cofinal subspectrum of $\bigvee_{\alpha\in A} E^\alpha$.

Now, let's show that

$$\{- \circ i_\alpha\}_{\alpha\in A} : Hom(\bigvee_{\alpha\in A} E^\alpha, F) \cong \prod_{\alpha\in A} Hom(E^\alpha, F).$$

- *surjective:* Let $\prod_{\alpha\in A}[\overline{E}^\alpha, f_\alpha] \in \prod_{\alpha\in A} Hom(E^\alpha, F)$. Then, consider $[\bigvee_{\alpha\in A} \overline{E}^\alpha, (f_\alpha)_{\alpha\in A}]$.

$$\{- \circ i_\alpha\}_{\alpha\in A}\Big([\bigvee_{\alpha\in A} \overline{E}^\alpha, (f_\alpha)_{\alpha\in A}]\Big) = \prod_{\alpha\in A}[\overline{E}^\alpha, f_\alpha]$$

- *injective:* Consider $[\overline{E_1}, f], [\overline{E_2}, g]$ be such that

$$\{- \circ i_\alpha\}_{\alpha\in A}\big([\overline{E_1}, f]\big) = \prod_{\alpha\in A}[\overline{E}_1 \cap E^\alpha, f \circ i_\alpha] = \prod_{\alpha\in A}[\overline{E}_2 \cap E^\alpha, g \circ i_\alpha] = \{- \circ i_\alpha\}_{\alpha\in A}\big([\overline{E_2}, g]\big)$$

  Then consider $[E_3^\alpha, h_\alpha]$ under $[\overline{E}_1 \cap E^\alpha, f \circ i_\alpha]$ and $[\overline{E}_2 \cap E^\alpha, g \circ i_\alpha]$. We have

$$[\bigvee_{\alpha\in A} E_3^\alpha, (h_\alpha)_{\alpha\in A}] \text{ is under } [\overline{E}_1, f] \text{ and } [\overline{E}_2, g], \text{i.e. } [\overline{E}_1, f] = [\overline{E}_2, g]$$

For the homotopy part, we have to use that $\left(\bigvee_{\alpha\in A} E^\alpha\right) \wedge I^+ = \bigvee_{\alpha\in A} E^\alpha \wedge I^+$ and that $\overline{E}$ is cofinal in $E \wedge I^+ \iff \overline{E} = E' \wedge I^+$ with E' cofinal in E.

Then, an homotopy between $[E_1, f]$ and $[E_2, g]$, with E_1, E_2 cofinal subspectra of $\bigvee_{\alpha\in A} E^\alpha$, given by $[E_3 \wedge I^+, h]$ is given one to one with $\prod_{\alpha\in A}[(E_3 \cap E^\alpha) \wedge I^+, h \circ i_\alpha]$.

□ 2.1.28

Now, we want to prove the spectral version of the Whitehead theorem 2.1.32. To do so, we will need the following technical lemmas.

Lemma 2.1.29.
Consider the following commutative diagram of spectra and functions:

$$\begin{array}{ccc} F & \xrightarrow{f} & E \\ \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle h} \\ A & \xrightarrow{\iota} & B \end{array}$$

with f being a weak homotopy equivalence. Then, we can find a cofinal subspectra $B' \subset B$ with functions $h' : B' \to F$ and $k : B' \wedge I^+ \to D$ such that:

1. $A \subset B'$.
2. $h'|_A = g$.
3. $k \circ \iota_0 = f \circ h'$, $k \circ \iota_1 = h|_{B'}$.
4. $k(a, t) = k(a, s)\ \forall a \in A, t, s \in I$.

Proof of Lemma 2.1.29:
Let
$$\mathbf{T} = \Big\{(A', h', k')|\ A' \text{ is a subspectrum and it follows property 1 to 4 }\Big\}$$
and we work on $\mathbf{T}$ as a poset using
$$(A', h', k') \leqslant (A'', h'', k'') \iff A' \subset A'', h''|_{A'} = h', k''|_{A'\wedge I^+} = k'$$
Consider C a chain in the poset $\mathbf{T}$. Let's show it has a supremum.
We define $M = \Big(\bigcup_{D\in C} D, \bigcup_{D\in C} h_D, \bigcup_{D\in C} k_D\Big)$. It is easy to see that M still preserve all 4 properties and by construction, $\forall D \in C, D \leqslant M$
Thus, $(\mathbf{T}, \leqslant)$ is an inductive poset. Using Zorn's Lemma, we get $\exists\ (B', h', k)$ maximal in $\mathbf{T}$.

Now, let's show B' cofinal. Consider a cell $e \in B_n$ and $S(e)$. By its definition, we have that $S(e) \cong \Sigma^n\Sigma^\infty S^r$. Then, $S(e)$ can be seen as a cofinal subset of $\Sigma^m\mathbb{S}$ (with $m = n + r$).
Then, consider $[S(e), h] \in \pi_m(E)$. using the fact that f is a function and a weak equivalence, we get that $\exists[U, \mu] \in \pi_m(F)$ such that $[U, f \circ \mu] \sim_{Hom} [S(e), h]$. Using 2.1.10, we thus have $U \cap S(e) = W$ cofinal in $S(e)$.

Thus, $\exists w$ such that $\Sigma'^{w} e \in W_{n+w}$. Note that all this construction fall back on g if we end up in A, and that thanks to $f \circ g = h|_A$.
If $\forall i \in \mathbb{N}, \Sigma'^{i}(e) \notin B'$, then du to the nature of W, we get that $W \cap B' = \{*\}$. Thus, $(B' \sqcup W, h' \sqcup \mu, W, k \sqcup \kappa) \in \mathbf{T} \geqslant (B', h', k)$, which contradict the maximality of B'.

Thus, B' cofinal.

□ 2.1.29

Corollary 2.1.30.
Consider the following commutative diagram of spectra and maps with f a weak equivalence:

$$\begin{array}{ccc} F & \xrightarrow{f} & E \\ {\scriptstyle g}\uparrow & \nwarrow{\scriptstyle k} & \uparrow{\scriptstyle h} \\ A & \xrightarrow{\iota} & B \end{array}$$

Then, $\exists k \in Hom(B, F)$, $k|_A = g$ such that $f \circ k \sim_{Hom} h$

Proof of Corollary 2.1.30:
Let $f = [F', f'], h = [B', h'], g = [A', g']$ with $g'(A') \subset F'$ and $A' \subset B'$ (by using lemma 2.1.12). Because $f' = f \circ \iota'_F$ with ι'_F its own inverse in $\mathbf{Sp}$, we get that $f'_* = f_* \circ \iota_*$ and thus f' is a weak equivalence function between F' and E. Then, we get the following commutative diagram of spectra and functions:

$$\begin{array}{ccc} F' & \xrightarrow{f'} & E \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle h'} \\ A' & \xrightarrow{\iota} & B' \end{array}$$

Using previous lemma 2.1.29, we get $(\overline{B}, \overline{h}, \overline{k})$ with $\overline{B}$ cofinal in B' and thus in B. We define $k = [\overline{B}, \overline{h}]$ and the homotopy between $f \circ k = [\overline{B}, f \circ \overline{h}]$ and $h = [\overline{B}, h'|_{\overline{B}}]$ is given by $H = [\overline{B} \wedge I^+, \overline{k}]$.

□ 2.1.30

Corollary 2.1.31.
Let $E, F \in \mathbf{Sp}$ and $f \in Hom(E, F)$ such that f is a weak homotopy equivalence. Then $\forall G \in \mathbf{Sp}$

$$f_* : [G, E] \cong [G, F].$$

Proof of Corollary 2.1.31:

- *surjectivity*: Consider the map $h : G \to F$ and the spectra $* = \Sigma^\infty *$. composed only of based point. Then, we construct the following commutative maps diagram:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle h} \\ * & \xrightarrow{\iota} & G \end{array}$$

  Then, using corollary 2.1.29, we get $k : G \to E$ such that $f \circ k \sim_{Hom} h$.

- *injectivity*: Consider the maps $g_1, g_2 : G \to E$ define as $g_i = [G_i, g_i']$ such that $f \circ g_1 \sim_{Hom} f \circ g_2$ using the map $h : G \wedge I^+ \to F$. We define $A = G \wedge \{0,1\}^+$ and the map $g : A \to E$ defined as $\left[G_1 \wedge \{0\} \sqcup G_2 \wedge \{1\}, g_1' \sqcup g_2'\right]$.

  Then, we construct the following commutative maps diagram:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle h} \\ A & \xrightarrow{\iota} & G \wedge I^+ \end{array}$$

  By using corollary 2.1.29, we get $k : G \wedge I^+ \to E$ such that $k|_A = g$. Thus, k is an homotopy between g_1 and g_2.

□ 2.1.31

Theorem 2.1.32 (Whitehead's spectral theorem)**.**
Let $E, F \in Ob(\mathbf{Sp})$, $f \in Hom(E, F)$. Then:

$$f \textit{ is an homotopy equivalence} \iff f \textit{ is a weak homotopy equivalence.}$$

Proof of Theorem 2.1.32:

- $\underline{\Rightarrow}$: Already done on definition 2.1.24.

- $\underline{\Leftarrow}$: Consider those 2 bijections

$$f_*[E, E] \to [E, F]$$

$$f_*[F,E] \to [F,F]$$

and let $g \in \mathbf{Sp}(F,G)$ such that $[f \circ g] = f_*[g] = [id_F]$ (that exists by surjectivity).

Furthermore, consider $g \circ f : E \to E$. We have $f_*[g \circ f] = [(f \circ g) \circ f] = [id_Y \circ f] = [f] = f_*[id_E]$ and by injectivity, this gives us that $[g \circ f] = [id_E]$.

Thus, f is indeed an homotopy equivalence with inverse g.

□ 2.1.32

We are now free to define the category of homotopy on spectra.

Definition 2.1.33 (Homotopy spectra category)**.**
*We define the **homotopy spectra category**:* **HoSp**

$$Ob(\mathbf{HoSp}) = Ob(\mathbf{Sp})$$

$$\mathbf{HoSp}(E,F) = [E,F]$$

f is an isomorphism in **HoSp** $\iff$ *f is a weak homotopy equivalence.*

Here are a few interesting properties of homotopy on spectra.

Proposition 2.1.34.
Let $X, Y \in Ob(\mathbf{HoCW})$, $E, F \in Ob(\mathbf{HoSp})$. Then:

1. *if X is weakly homotopically equivalent to Y, then $E \wedge X \sim_{Hom} E \wedge Y$ and $\Sigma^\infty X \sim_{Hom} \Sigma^\infty Y$.*

2. *if $E \sim_{Hom} F$, then $E \wedge X \sim_{Hom} F \wedge X$*

Proof of Proposition 2.1.34:

1. Let $f : X \to Y$ be a weak homotopy equivalence. Then, by using whitehead's theorem 1.4.12, it is an homotopy equivalence. Consider $g : Y \to X$, h_1 the homotopy between $f \circ g$ and id_Y and h_2 the homotopy between $g \circ f$and id_X.

   for $E \wedge X$, see that $(E \wedge X) \wedge I^+ = E \wedge (X \wedge I^+)$. Therefore, we get that $id_E \wedge h_1$ is an homology between $(id_E \wedge f) \circ (id_E \wedge g) = (id_E \wedge f \circ g)$ and $id_{E \wedge I^+}$. It is similar for h_2.

   $\Sigma^\infty X$ comes because $\Sigma^\infty X = \mathbb{S} \wedge X$.

2. Let $f : E \to F$, $g : F \to E$ be the homotopy equivalence with h_1 the homotopy between $f \circ g$ and id_F and h_2 the homotopy between $g \circ f$and id_E. Then, the maps $f \wedge id_X$ and $g \wedge id_Y$ are homotopy equivalence with homotopy maps $h_1 \wedge id_X$ and $h_2 \wedge id_X$.

$\square$ 2.1.34

Now, we want to show the homotopy extension on spectra, similarly to 1.4.16. We need the following lemma.

Lemma 2.1.35.
Let $E, H \in \mathbf{Sp}$, F be a subspectrum E and G be a cofinal subspectrum of $E \wedge \{0\}^+ \cup F \wedge I^+$. Given a function $g : G \to H$, we can find K a cofinal subspectrum of $E \wedge I^+$ containing G and an extension of g named $k : K \to H$

Proof of Lemma 2.1.35:
We will construct the K^n layers and $k|_{K^n} \to H$ by induction.
We have that $K^0 = \{*\}, k|_{K^0} = *$.
Then, suppose we have defined K^n layers and $k|_{K^n} \to H$ such that K^n cofinal in $E^n \wedge I^+$, $g^n \subset K^n$ and $k|_{G^n} = g|_{G^n}$.
For every cell $e = \{e_m, \Sigma' e_m, \cdots\}$ of $E^{n+1} \backslash (E^n \cup F^{n+1})$, we can find a N large enough using cofinality such that $\Sigma'^N e_m \wedge I^+$ is attach to K^n_{m+N} and g is defined on $\Sigma'^N e_m \wedge \{0\}^+$. Then, we have a map

$$(g_{m+N} \cup k_{m+N}) : \Sigma'^N e_m \wedge \{0\}^+ \cup \Sigma'^N \partial e_m \wedge I^+ \to H_{m+N}$$

Then, by using the homotopy extension property on CW complexes 1.4.16, we get that we can extend it into a map

$$v_e : \Sigma'^N e_m \wedge I^+ \to H_{m+N}$$

Thus, we add all $\Sigma'^N e_m \wedge I^+$ and its suspensions to $K^n \cup G^{n+1}$, getting K^{n+1} and $k|_K^{n+1}$ using v_e. We have that by construction that K^{n+1} is cofinal in $E_{n+1} \wedge I^+$, that $G_{n+1} \subset K^{n+1}$ and $k|_{G^{n+1}} = g|_{G^{n+1}}$.
Thus, by taking $K = \bigcup_n K^n$ and $k = \bigcup_n k|_{K^n}$, we get our desired result.

$\square$ 2.1.35

Corollary 2.1.36 (Homotopy extention on spectra)**.**
Let $E, H \in Ob(\mathbf{Sp})$, $h \in \mathbf{Sp}(E, H)$. Let F be a subspectrum of E and $g : F \to H$ a map. Then, if we have that $h|_F \sim_{Hom} g$, then $\exists \overline{g} : E \to H$ such that

$$h \sim_{Hom} \overline{g},\ \overline{g}|_F = g$$

Proof of Corollary 2.1.36:
Let $h = [E', h']$ and $[U, f'] = f$ with f the homotopy between $h|_F \sim_{Hom} g$. Then, we define $G = E' \wedge \{0\} \cup U$ a cofinal subspectrum of $E \wedge \{0\} \cup F \wedge I^+$. Then, using previous lemma2.1.35 on the function

$$h' \cup f' : G \to H$$

we get that a map $k : E \wedge I^+ \to H$ $k = [K, k']$ such that k' is an extension of $h' \cup f'$. We thus have our desired homotopy k between h and $k \circ i_1 = \overline{g}$ such that $k \circ i_1|_A = g$.

□ 2.1.36

2.1.3 Group structure on spectral homotopy sets

In this part, our goal is now to give further structure on $[E, F]$. To do so, here is a small property that can help us simplify some construction.

Proposition 2.1.37.
Let $E \in Ob(\mathbf{Sp})$ Then $\exists E' \in Ob(\mathbf{Sp})$ such that $E \sim_{Hom} E'$ and the maps $p_n : \Sigma E'_n \hookrightarrow E'_{n+1}$ are simple inclusions (i.e. $\Sigma E'_n \subset E'_{n+1}$)

Proof of Proposition 2.1.37:
Considering E as $\{E_n, p_n\}$. We use lemma 2.1.7 and get a spectra E'. We also get cellular maps $r_n : E'_n \to E_n$ that are homotopy equivalence (with inverse i_n) using h_n. Then, we define the spectral function :

$$r : E' \to E$$

$$r = \{r_n\}$$

because $r_{n+1}|_{\Sigma E'_n} = p_n \circ \Sigma r_n$, this is indeed a function.
Now, knowing that $r_{k*} : \pi_q(E'_k) \cong \pi_q(E_k)$ and that $\pi_n(F) \cong \operatorname{colim}_k \pi_{n+k}(F_k)$ we get that $\forall n \in \mathbb{Z}$, this diagram

$$\begin{array}{ccc} \pi_n(E') & \xrightarrow{\quad r_* \quad} & \pi_n(E) \\ \Big\| \alpha & & \alpha \Big\| \\ \operatorname{colim}_k \pi_{n+k}(E'_k) & \overset{\{(r_k)_*\}}{=\!=\!=} & \operatorname{colim}_k \pi_{n+k}(E_k) \end{array}$$

commute, thus, r is a weak homotopy equivalence. Using theorem 2.1.32, we get that r is an homotopy equivalence between E and E'

□ 2.1.37

We will now show a very important theorem in spectra. If it looks similar to 1.1.11, it is far more complex to prove.

Theorem 2.1.38.
Let $E \in Ob(\mathbf{HoSp})$

$$\Sigma E \sim_{Hom} E \wedge S^1$$

Proof of Theorem 2.1.38:
Using proposition 2.1.37, we get that $E \sim_{Hom} E'$, with

$$E'_n = E_n \wedge \{n\}^+ \cup \bigsqcup_{k<n} \Sigma^{n-k} E_k \wedge [k, k+1]^+_{/\sim}, n \in \mathbb{Z}$$

$[x, k+1] \sim [\Sigma^{n-k-1} p_k(x), k+1]$.
We will construct an homotopy equivalence $f : E' \wedge S^1 \to \Sigma E$.
We define the following functions:

- $\nu' : S^1 \cong S^1$ given in example 1.2.5 and using that $S^1 = \Sigma(\{1\}^+)$
- $\alpha : I^2 \cong D^2$ the homeomorphism obtained by centering and then extending radially, thus insuring that $\alpha(\partial I^2) \cong S^1$.
- $K : D^2 \times I \to D^2$ the homotopy given by

  $$K(x, y, t) = \Big(x \cos(\pi t/2) - y \sin(\pi t/2), x \sin(\pi t/2) + y \cos(\pi t/2)\Big)$$

  $K(x, y, 0) = (x, y), K(x, y, 1) = (-y, x)$. We furthermore have that $K(S^1) \to S^1$
- $\alpha^{-1} \circ K \circ \alpha$ is an homotopy between I^2 and itself. $K(x, y, 0) = (x, y)$, $K(x, y, 0) = (1-y, x)$. Using the fact that $S^1 \wedge S^1 = I^2/(\partial I \times I \cup I \times \partial I)$, we define

  $$H : (S^1 \wedge S^1) \wedge I \to S^1 \wedge S^1$$

  $$H = q \circ K \text{ with } q \text{ the quotient map}$$

  Then, we have $H \circ i_0 = id_{S^1 \wedge S^1}$ and $H \circ i_1(x, y) = [\nu'(y), x]$.

We consider $E'_n \wedge S^1$ as $(E_n \wedge \{n\}^+ \wedge S^1) \cup \bigsqcup_{k<n} (S^{n-k-1} \wedge S^1 \wedge E_k \wedge [k, k+1]^+ \wedge S^1)_{/\sim}$
Now, consider

$$f_n : E'_n \wedge S^1 \to E_{n+1}$$

$$f_n([\xi, n, y]) = p_n([\nu'^n(y), \xi]) \in E_{n+1} \text{ with } [\nu'^n(y), \xi] \in S^1 \wedge E_n = \Sigma E_n$$

$$f_n([s, x, \xi, m+t, y]) = (p_n \circ \Sigma p_{n-1} \circ \cdots \circ \Sigma^{n-m} p_m)\Big[s, H\big(x, \nu'^m(y), t\big), \xi\Big] \in E_{n+1}$$

$$\text{with } \Big[s, H\big(x, \nu'^m(y), t\big), \xi\Big] \in \Sigma^{n-m+1} E_m$$

$$\text{with } s \in S^{n-m-1}, x, y \in S^1, \xi \in E_m, t \in [0, 1]$$

After looking at it for a while, we see that they are continuous and that

$$p_{n+1} \circ \Sigma f_n([s, x, \xi, m+t, y]) = (p_{n+1} \circ \Sigma p_n \circ \Sigma^2 p_{n-1} \circ \cdots \circ \Sigma^{n-m+1} p_m)\Big[\bar{s}, H\big(x, \nu'^m(y), t\big), \xi\Big] =$$

$$= f_{n+1}|_{\Sigma E'_n \wedge S^1}[s, x, \xi, m+t, y]$$

$$\begin{array}{rcl} p_{n+1} \circ \Sigma f_n([\xi, n, y]) & = & (p_{n+1} \circ \Sigma p_n)[x, \nu'^n(y), \xi] \\ & = & (p_{n+1} \circ \Sigma p_n)[H(x, \nu'^n(y), 0), \xi] \\ & = & f_{n+1}|_{\Sigma E'_n \wedge S^1}[x, \xi, n, y] \end{array}$$

Thus, $\{f_n\}$ define a function $f : E' \wedge S^1 \to \Sigma E$. Let's now show that this is a weak homotopy equivalence.
We consider the following maps

$$g_n : \Sigma E_n = E_n \wedge S^1 \to E'_n \wedge S^1$$

$$g_n([x, \xi]) = [\xi, n, \nu'^n(x)]$$

with $\xi \in E_n$, $x \in S^1$. Then, we have that $f_n \circ (g_n \circ p_n^{-1}) = id_{\Sigma' E_n}$. Indeed

$$f_n \circ (g_n \circ p_n^{-1})(p_n(x, \xi)) = f_n \circ g_n([x, \xi]) = f_n([\xi, n, \nu'^n(x)]) = p_n(\nu'^{2n}(x), \xi) = p_n(x, \xi)$$

Furthermore, we have that $(g_n \circ p_n^{-1}) \circ f_n \sim_{Hom} id_{E_n \wedge S^1}$. We have that $(g_n \circ p_n^{-1}) \circ f_n|_{E_n \wedge \{n\}} = id_{E_n \wedge \{n\}}$. Thus, using an homotopy we defined on 2.1.7, we get

$$(g_n \circ p_n^{-1}) \circ f_n \circ (H \wedge id_{S^1}) : E'_n \wedge I^+ \wedge S^1 \to E'_n \wedge S^1$$

$$(g_n \circ p_n^{-1}) \circ f_n \circ (H \wedge id_{S^1}) \circ i_0 = (g_n \circ p_n^{-1}) \circ f_n \circ id_{E'_n \wedge S^1} = (g_n \circ p_n^{-1}) \circ f_n$$

$$(g_n \circ p_n^{-1}) \circ f_n \circ (H \wedge id_{S^1}) \circ i_1 = (g_n \circ p_n^{-1}) \circ f_n \circ (i_n \circ r_n) \wedge id_{S^1} = (i_n \circ r_n) \wedge id_{S^1} \sim_{Hom} id_{E'_n \wedge S^1}$$

Using this, we get that

$$f_* : \pi_k(E'_n \wedge S^1) \cong \pi_k(\Sigma' E_n)$$

Then, we consider the spectra $\Sigma' E$:

$$(\Sigma' E)_n = \Sigma'(E_n) \subset E_{n+1}$$

$p'_n : \Sigma(\Sigma' E)_n \hookrightarrow (\Sigma' E)_{n+1}$is given by: $p_{n+1} \circ \Sigma|_{\Sigma' E_n}$

$$p_{n+1}(\Sigma \Sigma' E_n) \cong \Sigma'(\Sigma' E_n) \subset \Sigma'(E_{n+1})$$

$\Sigma' E$ is a cofinal subspectra of ΣE. indeed $e \in (\Sigma E)_n = E_{n+1}$, then $\Sigma'(e) \in (\Sigma' E)_{n+1}$.
Using this result, we consider the following commutative diagram:

$$\begin{array}{ccc} \pi_n(E' \wedge S^1) & \xrightarrow{\quad f_* \quad} & \pi_n(\Sigma' E) \\ \| \ \alpha & & \alpha \ \| \\ \mathrm{colim}_k \pi_{n+k}\big((E' \wedge S^1)_k\big) & \overset{\{(f_k)_*\}}{=\!=\!=} & \mathrm{colim}_k \pi_{n+k}((\Sigma' E)_k) \end{array}$$

This induce that f is a weak homotopy equivalence. Thus, by theorem 2.1.32, f gives us that $E'_n \wedge S^1 \sim_{Hom} \Sigma' E$ but because $\Sigma' E$ cofinal in ΣE we finally get that:

$$E \wedge S^1 \sim_{Hom} E' \wedge S^1 \sim_{Hom} \Sigma' E \cong \Sigma E$$

□ 2.1.38

The previous theorem allows us to give spectral homotopy set a group structure.

Corollary 2.1.39.
*Let $E, F \in$ **HoSp**. We can give each set $[E, F]$ the structure of an abelian group so that composition is bilinear.*

Proof of Corollary 2.1.39:
Since Σ is an invertible, $[E, F] \cong [\Sigma E, \Sigma F]$. Thus, using 2.1.38 $[\Sigma E, \Sigma F] \cong [E \wedge S^1, F \wedge S^1]$.
We define the function σ:

$$\sigma : [E, F] \cong [E \wedge S^1, F \wedge S^1]$$

$$\sigma([f]) = [f \wedge id_{S^1}]$$

Consider $\sigma^2 : [E, F] \cong [E \wedge S^2, F \wedge S^2]$. We construct a co-multiplication on S^2 with base point x_0 using our example 1.2.5.

$$\mu : S^2 \cong \Sigma S^1 \xrightarrow{\mu'} \Sigma S^1 \vee \Sigma S^1 \cong S^2 \vee S^2$$

We also define the inverse map ν:

$$\nu :\cong \Sigma S^1 \xrightarrow{\nu'} \Sigma S^1 \cong S^2$$

Thus, we define $\overline{\mu}$ for spectra

$$\overline{\mu} : E \wedge S^2 \xrightarrow{id_E \wedge \mu} E \wedge (S^2 \vee S^2) \cong (E \wedge S^2) \vee (E \wedge S^2)$$

$$\overline{\nu} : E \wedge S^2 \xrightarrow{id_E \wedge \nu} E \wedge S^2$$

This gives $E \wedge S^2$ the structure of H-commutative cogroup on $E \wedge S^2$. Indeed:

$$\big(id_{E \wedge S^2}, \{*\}\big) \circ \overline{\mu} = \big(id_E \wedge id_{S^2}, \{*\}\big) \circ (id_E \wedge \mu) = id_E \wedge \big(id_{S^2}, \{*\}\big) \circ \mu \sim_{Hom} id_E \wedge id_{S^2} = id_{E \wedge S^2}$$

$$\big(\{*\}, id_{E \wedge S^2}\big) \circ \overline{\mu} = id_E \wedge \big(\{*\}, id_{S^2}\big) \circ \mu \sim_{Hom} id_E \wedge id_{S^2} = id_{E \wedge S^2}$$

$$\big(id_{E \wedge S^2} \vee \overline{\mu}\big) \circ \overline{\mu} = id_E \wedge \big(id_{S^2} \vee \mu\big) \circ \mu \sim_{Hom} id_E \wedge \big(\mu \vee id_{S^2}\big) \circ \mu = \big(\overline{\mu}, id_{E \wedge S^2}\big) \circ \overline{\mu}$$

$$\big(\overline{\nu}, \{*\}\big) \circ \overline{\mu} = id_E \wedge \big(\nu, \{*\}\big) \circ \mu \sim_{Hom} id_E \wedge \{*\} = \{*\}$$

$$\big(\{*\}, \overline{\nu}\big) \circ \overline{\mu} = id_E \wedge \big(\{*\}, \nu\big) \circ \mu \sim_{Hom} id_E \wedge \{*\} = \{*\}$$

$$\overline{\mu} \circ T = id_E \wedge (\mu \circ T) \sim_{Hom} id_E \wedge \mu = \overline{\mu}$$

Now, we can finally define our composition law on $[E, F]$:

$$* : [E, F] \times [E, F] \to [E, F]$$
$$[f] * [g] = [\sigma^{-2} \circ \underline{\mu}\Big(\sigma^2([f]), \sigma^2([g])\Big)]$$

$$\underline{\mu} : [E \wedge S^2, F \wedge S^2] \times [E \wedge S^2, F \wedge S^2] \to [E \wedge S^2, F \wedge S^2]$$

$\underline{\mu}([f], [g])$ is define as the homotopy class of the following map:

$$E \wedge S^2 \xrightarrow{\overline{\mu}} (E \wedge S^2) \vee (E \wedge S^2) \xrightarrow{f \vee g} (F \wedge S^2) \vee (F \wedge S^2) \xrightarrow{\Delta} F \wedge S^2$$

with $\Delta(x, *) = x = (*, x)$
First, we see that $\Delta \circ - \vee - = (-, -)$ as define in 1.2.4 and that it suffice to work on $\underline{\mu}$ to show our result.

- *well defined*: Let $f, g : E \wedge S^2 \to F \wedge S^2$, $f \sim_{Hom} f'$ using h_f, $g \sim_{Hom} g'$ using h_g. Then, let's show $\underline{\mu}(f, g) \sim_{Hom} \underline{\mu}(f', g')$. Because $h_f : E \wedge S^2 \wedge I^+ \to F \wedge S^2$, consider

$$\underline{\mu}(h_f, h_g) = \Delta \circ (h_f \vee h_g) \circ \overline{\mu}.$$

  Because $\overline{\mu} = id_{E \wedge I^+} \wedge \mu$ we have that $\overline{\mu} \circ i_0 = i_0 \circ \overline{\mu}$. This imply

$$\underline{\mu}(h_f, h_g) \circ i_0 = \underline{\mu}(h_f \circ i_0, h_g \circ i_0) = \underline{\mu}(f, g)$$

  Thus $\underline{\mu}(h_f, h_g)$ is our homotopy between $\underline{\mu}(f, g)$ and $\underline{\mu}(f', g')$

- *associative*: Let $f, g, h \in \mathbf{Sp}(E \wedge S^2, F \wedge S^2)$. We have to show that

$$\underline{\mu}\big(f, \underline{\mu}(g, h)\big) \sim_{Hom} \underline{\mu}\big(\underline{\mu}(f, g), h\big)$$

$$\begin{array}{rcl}
\underline{\mu}\big(f, \underline{\mu}(g, h)\big) & = & \Delta \circ f \vee \big(\Delta \circ (g \vee h) \circ \overline{\mu}\big) \circ \overline{\mu} \\
& = & \Delta \circ \Big((f \circ id) \vee (g, h) \circ \overline{\mu}\Big) \circ \overline{\mu} \\
& = & \Delta \circ \big(f \vee (g \vee h)\big) \circ (id \vee \overline{\mu}) \circ \overline{\mu} \\
& \sim_{Hom} & \Delta \circ \big((f \vee g) \vee h)\big) \circ (\overline{\mu} \vee id) \circ \overline{\mu} \\
& = & \Delta \circ \Big(\big((f, g) \circ \overline{\mu}\big) \vee h \circ id\Big) \circ \overline{\mu} \\
& = & \Delta \circ (\Delta \circ f \vee g \circ \overline{\mu}) \vee h \circ \overline{\mu} \\
& = & \underline{\mu}\big(\underline{\mu}(f, g), h\big)
\end{array}$$

- *neutral element*: Consider $\{*\} : E \wedge S^2 \to F \wedge S^2$ the map that send everything into the base point. Then, $\forall f \in \mathbf{Sp}(E, F)$

$$\begin{array}{rcl}
\underline{\mu}(\{*\}, f) & = & \Delta \circ \{*\} \vee f \circ \overline{\mu} \\
& = & (\{*\}, f) \circ \overline{\mu} \\
& = & f \circ (\{*\}, id_{E \wedge S^2}) \circ \overline{\mu} \\
& \sim_{Hom} & f \circ id_{E \wedge S^2} = f
\end{array}$$

- *inverse*: $\forall f \in \mathbf{Sp}(E,F)$, consider $f \circ \overline{\nu} \in \mathbf{Sp}(E,F)$. Then

$$\begin{array}{rcl} \underline{\mu}(f \circ \overline{\nu}, f) & = & \Delta \circ (f \circ \overline{\nu} \vee f) \circ \overline{\mu} \\ & = & (f \circ \overline{\nu}, f) \circ \overline{\mu} \\ & = & f \circ (\overline{\nu}, id_{E \wedge S^2}) \circ \overline{\mu} \\ & \sim_{Hom} & f \circ \{*\} = \{*\} \end{array}$$

- *abelian*: $\forall f, g \in \mathbf{Sp}(E,F)$

$$\underline{\mu}(f,g) = \Delta \circ (f \vee g) \circ \overline{\mu} \sim_{Hom} \Delta \circ (f \vee g) \circ \overline{\mu} \circ T = \Delta \circ (g \vee f) \circ \overline{\mu} = \underline{\mu}(g,f)$$

- *composition is a bilinear map*: Indeed consider the composition map:

$$\circ : [E,F] \times [F,G] \to [E,G]$$

$$[f] \circ [g] = [g \circ f]$$

Then, we have that:

  - $[\{*\}] \circ [g] = [g \circ \{*\}] = [\{*\}] = [\{*\} \circ f] = [f] \circ [\{*\}]$
  - $[\underline{\mu}(f,g)] \circ [h] = [h \circ \underline{\mu}(f,g)] = [h \circ (f,g) \circ \overline{\mu}] = [(h \circ f, h \circ g) \circ \overline{\mu}] = [\underline{\mu}(h \circ f, h \circ g)]$
  - Consider $h : E \wedge S^2 \to F \wedge S^2$. Then , using $(id_{F \wedge S^2}, *) \circ \overline{\mu} \circ h \sim_{Hom} id_{F \wedge S^2} \circ h = h$, we get that

$$\overline{\mu} \circ h \sim_{Hom} (h \vee h) \circ \overline{\mu}$$

Thus:

$$[h] \circ [\underline{\mu}(f,g)] = [\underline{\mu}(f,g) \circ h] = [(f,g) \circ \overline{\mu} \circ h] = [(f,g) \circ (h \vee h) \circ \overline{\mu}] =$$

$$= [(f \circ h, g \circ h) \circ \overline{\mu}] = [\underline{\mu}(f \circ h, g \circ h)]$$

□ 2.1.39

Note that the last part gives us that pullback and pushforward are group morphism.

2.1.4 Induced homology and cohomology

Now, we want to show that homotopy sets have an exact sequence property on cofibres 2.1.44. To do such a thing, we will need to define cofibres on spectra and to prove a fews intermediary lemmas.

Definition 2.1.40 (Spectral cofibres sequence)**.**
Let $f \in \mathbf{Sp}(E, F)$. We call the following sequence

$$E \xrightarrow{f} F \xrightarrow{j} F \cup_f CE$$

the ***special cofibre sequence****.*
We define a ***general cofibre sequence*** *a sequence*

$$G \xrightarrow{g} H \xrightarrow{h} K$$

such that $\exists E, F \in \mathbf{Sp}$ and $f \in \mathbf{Sp}(E, F)$ and the following diagram commute up to homotopy.

$$\begin{array}{ccccc} G & \xrightarrow{g} & H & \xrightarrow{h} & K \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \sigma \\ E & \xrightarrow{f} & F & \xrightarrow{j} & F \cup_f CE \end{array}$$

with α, β, σ homotopy equivalence.

Proposition 2.1.41.
Given the following sequence:

$$E \xrightarrow{f} F \xrightarrow{j} F \cup_f CE \xrightarrow{k'} E \wedge S^1 \xrightarrow{f \wedge id} F \wedge S^1 \to \cdots$$

with $k'_n|_{F_n} = *, k'_n|_{C(E_n)\backslash[E_n,1]} = id$
Each pair of consecutive maps forms a cofibre sequence.

Proof of Proposition 2.1.41: [Swi17] 8.29 and 2.39. This is a straightforward result we won't have time to prove. Note that it also exists of CW-complexes. □ 2.1.41

Now, we give a remark that will help us simplify some of our later proofs.

Remark 2.1.42.
We can assume that f (or g in the general case) is an inclusion. Indeed, Let $f = [E', f']$, let $M_f = F \cup_{f'} E' \wedge I^+$ [4]. Then, using the rectraction map of E' into F given by

$$\nu_n : (M_f)_n \to F_n$$

$$\nu_n|_{Y_n} = id, \nu[e, t] = f(t)$$

we get that this is an homotopy equivalence and furthermore, we have that $\Sigma' \nu_n = \nu_{n+1}|_{\Sigma'(M_f)_n}$. Thus, we have the following commutative homotopy diagram:

$$\begin{array}{ccccc}
E & \xrightarrow{f} & F & \xrightarrow{j} & F \cup_f CE \\
\downarrow \iota & & \downarrow \nu & & \downarrow \sigma \\
E' & \xrightarrow{\iota} & M_f & \xrightarrow{j} & M_f \cup_\iota CE'
\end{array}$$

with σ defined using the fact that $E' \wedge I^+ \cup_\iota CE' \sim_{Hom} C(E')$

Now, consider the following technical lemma

Lemma 2.1.43.
Given the following homotopy commutative diagram:

$$\begin{array}{ccccccc}
G & \xrightarrow{g} & H & \xrightarrow{h} & K & \xrightarrow{k} & G \wedge S^1 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \kappa & & \downarrow \alpha \wedge id \\
G' & \xrightarrow{g'} & H & \xrightarrow{h'} & K' & \xrightarrow{k'} & G' \wedge S^1
\end{array}$$

such that the rows are cofibre sequences. The, we can complete the diagram with κ.

Proof of Lemma 2.1.43:
wlog, we can assume that the rows are special cofibre sequences. Furthermore, we may assume that g is a standard inclusion. We carefully chose the representatives of $\beta = [B', \beta']$, $g' = [A', f']$ and $[A, \alpha'] = \alpha$ such that $g(A) \subset B$ and $\alpha'(A) \subset A'$. Then, we have the following homotopy commuting diagram.

[4] We have $(M_f)_n = F_n \cup_{f'} E'_n \wedge I^+$ with $[x, 1] \sim f'_n(x)$ for $x \in E'_n$.

$$\begin{array}{ccccccc}
A & \xrightarrow{g} & B & \xrightarrow{j} & B \cup_g CA & \longrightarrow & A \wedge S^1 \\
\downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \alpha' \wedge id} \\
A' & \xrightarrow{g'} & H' & \xrightarrow{j} & H' \cup_{f'} CA' & \longrightarrow & A' \wedge S^1
\end{array}$$

Then, using the homotopy extension lemma 2.1.36, we can construct

$$\beta'' : B \to H'$$

$$\beta'' \circ g = f' \circ \alpha', \beta' \sim_{Hom} \beta$$

We thus define κ

$$\kappa|_B = \beta'', \kappa|_{CA} = C(\alpha')$$

The following diagram commute strictly

$$\begin{array}{ccccccc}
A & \xrightarrow{g} & B & \xrightarrow{j} & B \cup_g CA & \longrightarrow & A \wedge S^1 \\
\downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta''} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \alpha' \wedge id} \\
A' & \xrightarrow{g'} & H' & \xrightarrow{j} & H' \cup_{f'} CA' & \longrightarrow & A' \wedge S^1
\end{array}$$

making the previous one commute up to homotopy.

□ 2.1.43

The previous lemma help us put in light this beautiful property of spectral homotopy theory.

Lemma 2.1.44.
Let $G \xrightarrow{g} H \xrightarrow{h} K$ *be a cofibre sequences. Then,* $\forall E \in \mathbf{Sp}$*, the following sequences:*

$$[E, G] \xrightarrow{g_*} [E, H] \xrightarrow{h_*} [E, K]$$

$$[G, E] \xleftarrow{g^*} [H, E] \xleftarrow{h^*} [K, E]$$

are exacts.

Proof of Lemma 2.1.44:

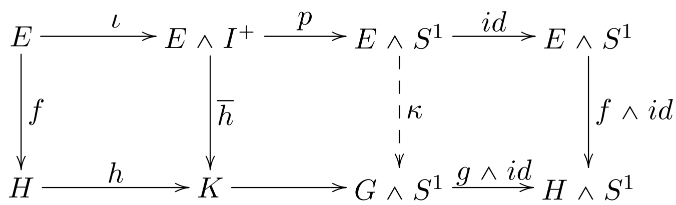
First, see that $h \circ g \sim_{Hom} 0$. Indeed $h \circ g \sim_{Hom} j \circ f : E \to F \cup_f CE$. With $j \circ f(e) = [f(e)] = [e, 1]$. Then, we define

$$h : E \wedge I^+ \to F \cup_f CE$$

$$h = [id_E \wedge t]$$

with $h \circ i_1 = j \circ f$, $h \circ i_0 = \{*\} = 0$.

- Because $h \circ g \sim_{Hom} 0$, we have that $(h \circ g)_* = h_* \circ g_* = 0_*$. Thus, $\mathrm{Im}(g_*) \subseteq \ker(h_*)$. Now, let $f : E \to H$ such that $h_*[f] = 0$. We consider the following homotopy commuting diagram

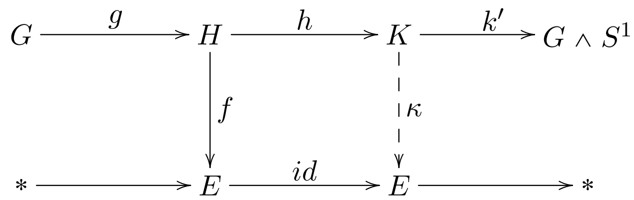


  with $\overline{h}$ the homotopy between 0 and $g \circ f$. Using lemma 2.1.43, we get κ such that $f \wedge id \sim_{Hom} (g \wedge id) \circ \kappa$. Using $\sigma : [E, F] \cong [E \wedge S^1, F \wedge S^1]$, we get $\kappa' : E \to G$, $\kappa = \kappa' \wedge id$. Thus,

$$(g \circ \kappa') \wedge id = (g \wedge id) \circ (\kappa' \wedge id) = (g \wedge id) \circ \kappa \sim_{Hom} f \wedge id$$

  Using the fact that σ is injective, we get that $[g \circ \kappa] = [f]$, proving our point.

- because $h \circ g \sim_{Hom} 0$, we have that $(h \circ g)^* = g^* \circ h^* = 0*$. Thus, $\mathrm{Im}(h^*) \subseteq \ker(g^*)$. Now, let $f : H \to E$ such that $g^*[f] = 0$. We consider the following homotopy commuting diagram



  We thus have using lemma 2.1.43 κ with $h^*([\kappa]) = \kappa \circ h \sim_{Hom} f$, proving our point.

□ 2.1.44

Knowing this lemma, we now can define induced (co)homology using spectra.

Definition 2.1.45 (Induced reduced homology and cohomology by spectra)**.**
Let $E \in Ob(\mathbf{HoSp})$. We define the ***reduced homology induced by*** *E:*

$$E_* : \mathbf{HoCW} \to \mathbf{Ab}$$

$$E_n(X) = \pi_n(E \wedge X) = [\Sigma^n \mathbb{S}, E \wedge X]$$

$$E_n(f) = (id_E \wedge f)_*$$

with σ_n : defined as

$$\sigma_n : E_n(X) = [\Sigma^n \mathbb{S}, E \wedge X] \cong [\Sigma^{n+1} \mathbb{S}, \Sigma E \wedge X] \cong [\Sigma^{n+1} \mathbb{S}, E \wedge S^1 \wedge X] \cong [\Sigma^{n+1} \mathbb{S}, E \wedge \Sigma X]$$

We also define the ***reduced cohomology induced by*** E*:*

$$E^* : \mathbf{HoCW} \to \mathbf{Ab}$$

$$E^n(X) = [\Sigma^\infty X, \Sigma^n E]$$

$$E^n(f) = (\Sigma^\infty f)^*$$

with σ^n *: defined as*

$$\sigma^n : E^{n+1}(\Sigma X) = [\Sigma^\infty \Sigma X, \Sigma^{n+1} E] \cong [\Sigma\Sigma^\infty X, \Sigma^{n+1} E] \cong [\Sigma^\infty X, \Sigma^n E]$$

using cofinality of $\Sigma^\infty \Sigma X$ *in* $\Sigma\Sigma^\infty X$

Verification of Definition 2.1.45:
Let's show that $\{E_*, \sigma_*\}$ and $\{E^*, \sigma^*\}$ are respectively a reduced homology and a reduced cohomology.
First, consider $X \cup_f CA$ a mapping cone of $f : A \to X$. Then

$$\Sigma^\infty(X \cup_f CA) = \Sigma^\infty(X) \cup_{\Sigma^\infty f} \Sigma^\infty CA = \Sigma^\infty X \cup_{\Sigma^\infty f} C\Sigma^\infty A.$$

Similarly,

$$E \wedge (X \cup_f CA) = E \wedge X \cup_f E \wedge CA = (E \wedge X) \cup_f C(E \wedge A)$$

Those results are useful because they allow us to transform mapping cone on CW complexes into mapping cone on spectra.

- $\forall f \in \mathbf{HoCW}(A, X)$. Then

$$E \wedge A \xrightarrow{id_E \wedge f} E \wedge X \xrightarrow{id_E \wedge j} E \wedge (X \cup_{id_E \wedge f} CA) = E \wedge (X \cup_f CA)$$

  is a cofibre sequence. Thus, using 2.1.44, we get the exact sequence

$$\pi_n(E \wedge A) \xrightarrow{(id_E \wedge f)_*} \pi_n(E \wedge X) \xrightarrow{(id_E \wedge j)_*} \pi_n\big(E \wedge (X \cup_f CA)\big)$$

- $\forall f \in \mathbf{HoCW}(A, X)$. Then

$$\Sigma^\infty A \xrightarrow{\Sigma^\infty f} \Sigma^\infty X \xrightarrow{\Sigma^\infty j} \Sigma^\infty(X \cup_f CA) = \Sigma^\infty X \cup_{\Sigma^\infty f} C\Sigma^\infty A$$

  is a cofibre sequence. Thus, using 2.1.44, we get the exact sequence

$$[\Sigma^\infty A, \Sigma^n E] \xleftarrow{(\Sigma^\infty f)^*} [\Sigma^\infty X, \Sigma^n E] \xleftarrow{(\Sigma^\infty j)^*} [\Sigma^\infty(X \cup_f CA), \Sigma^n E]$$

□ 2.1.45

Remark 2.1.46.

- See that thanks to it construction, the cohomology E^* follows the *wedge axiom*. Indeed, let $i_\alpha : X_\alpha \hookrightarrow \bigvee_{\alpha\in A} X_\alpha$. Then, using proposition 2.1.28, we get that

$$\{\Sigma^\infty i_\alpha\}_{\alpha\in A}[\bigvee_{\alpha\in A} \Sigma^\infty X_\alpha, F] \cong \prod_{\alpha\in A}[\Sigma^\infty X_\alpha, F]$$

  Now, using the fact that $\Sigma^\infty(X \vee Y) = (\Sigma^\infty X) \vee (\Sigma^\infty Y)$, we get that

$$\{(\Sigma^\infty i_\alpha)^*\}_{\alpha\in A} : E^n(\bigvee_{\alpha\in A} X_\alpha) = [\Sigma^\infty \bigvee_{\alpha\in A} X_\alpha, \Sigma^n E] \cong \prod_{\alpha\in A}[\Sigma^\infty X_\alpha, \Sigma^n E] = \prod_{\alpha\in A} E^n(X_\alpha)$$

- Furthermore, it is also apparent that E^* and E_* follows the *weak homotopy axiom*.

Also, using this definition, we can extend the notion of cohomology to spectra.

Definition 2.1.47 (Cohomology of a spectrum)**.**
*Let E, F be a spectrum. we define the E-**cohomology of** F as:*

$$E^n(F) = [F, \Sigma^n E]$$

Note that for $G \xrightarrow{f} H \xrightarrow{g} K$ a spectral cofibre sequence, using 2.1.44, we get the exact sequence

$$E^n(G) \xleftarrow{f^*} E^n(H) \xleftarrow{g^*} E^n(K)$$

Furthermore, if $\forall n \in \mathbb{N}, F_{-n} = *$, Then, we can see the E-cohomology of F as:

$$E^n(F) = \ lim_k \ E^{n+k}(F_k)$$

Verification of Definition 2.1.47:
Those two definitions are indeed the same. Indeed:

$$\begin{aligned} [F, \Sigma^n E] &= [\ \operatorname{colim}_k \Sigma^{-k}\Sigma^\infty F_k, \Sigma^n E] \\ &= \lim_k [\Sigma^{-k}\Sigma^\infty F_k, \Sigma^n E] \\ &= \lim_k [\Sigma^\infty F_k, \Sigma^{n+k} E] \\ &= \lim_k E^{k+n}(F_k) \end{aligned}$$

□ 2.1.47

2.2 Brown's Representation Theorem

Now that we know that spectra induce cohomology, we want to show the inverse, i.e. any cohomology with WHE and wedge axiom can be seen as given by a spectrum. We will dedicate the following section to that result, using Brown's representation theorem.

2.2.1 Main result

In all this section we work on contravariant functors

$$F : \mathbf{HoCW} \to \mathbf{Gp}$$

that they follows the following axioms

- Wedge W) :

$$(Fi_\alpha)_{\alpha \in A} : F(\bigvee_{\alpha \in A} X_\alpha) \cong \prod_{\alpha \in A} F(X_\alpha)$$

- Mayer-Vietoris MV) :

  For any CW-triad (X, A_1, A_2) (*i.e.* A_1, A_2 are subcomplex of $X = A_1 \cup A_2$) with $x_1 \in F(A_1), x_2 \in F(A_2)$ such that

$$F(i_{A_1 \cap A_2})(x_1) = x_1|_{A_1 \cap A_2} = x_2|_{A_1 \cap A_2} = F(i_{A_1 \cap A_2})(x_2)$$

  Then, $\exists y \in F(X)$ such that $y|_{A_1} = x_1, y|_{A_2} = x_2$

Proposition 2.2.1.
$\forall Y \in Ob(\mathbf{CW}), [-, Y] : \mathbf{HoCW} \to \mathbf{Set}$ *is a contravariant functor that satisfy W) and MV).*

Proof of Proposition 2.2.1:

- W): Let show $\{i_\alpha^*\}_{\alpha \in A}$ is surjective: For an element indexed by $\{[f_\alpha]\}_{\alpha \in A}$, we have $f : \vee_{\alpha \in A} X_\alpha \to Y$ defined with $f \circ i_\alpha = f_\alpha$. Then

$$\{i_\alpha^*\}_{\alpha \in A}(f) = \{[f \circ i_\alpha^*]\}_{\alpha \in A} = \{[f_\alpha]\}_{\alpha \in A}$$

  For injectivity: let $[f], [g]$ such that $\forall \alpha \in A, [f \circ i_\alpha] = [g \circ i_\alpha]$ using $H^\alpha : X_\alpha \wedge I^+$. Then, $H : (\bigvee_{\alpha \in A} X_\alpha) \wedge I^+ \to Y$ given by $H \circ i_\alpha = H^\alpha$ is the homotopy between $[f]$ and $[g]$.

- MV): Let $X = A_1 \cup A_2$ with $[f_1] \in [A_1, Y]$, $[f_2] \in [A_2, Y]$ such that $f_1|_{A_1 \cap A_2} \sim_{Hom} f_2|_{A_1 \cap A_2}$ using H. Then, du to the fact that the inclusion in CW-complex has the homotopy extension property, we get that there exists $\overline{H} : A_1 \wedge I^+ \to Y$, $\overline{H} \circ i_0 = f_1$, $\overline{H} \circ i_1 = f_1'$ with $f_1'|_{A_1 \cap A_2} = f_2|_{A_1 \cap A_2}$. Thus, we define

$$g : X \to Y$$

$$g(x) = \begin{cases} f_1'(x) & x \in A_1 \\ f_2(x) & x \in A_2 \end{cases}$$

$[g|_{A_1}] = [f_1], [g|_{A_2}] = [f_2]$.

□ 2.2.1

Note that because $[\Sigma -, Y] \cong [-, \Omega Y]$, we get that $\forall Y \in \mathbf{HoCW}$

$$[-, \Omega Y] : \mathbf{HoCW} \to \mathbf{Gp}$$

is a contravariant functor that follows W) and MV).
Now, we want to show 2.2.10. Every coming definitions and lemmas are needed preliminary results. First, let's compute the image of a singleton by F.

Proposition 2.2.2.
Let $*$ be the base point, F be a contravariant functor as defined earlier. Then

$$F(*) = 0$$

with 0 *the zero group.*

Proof of Proposition 2.2.2:
Using the fact that $F(\{x_0\}) = F(\{x_0\} \vee \{x_0\}) \cong F(\{x_0\}) \times F(\{x_0\})$, using map $F(a \to (a, a))$ we get that $|F(\{x_0\})| = 1$. Thus $F(\{x_0\}) = 0$

□ 2.2.2

Now, we see that follows some very similar property to the one of cohomology.

Lemma 2.2.3.
Let $X \in Ob(\mathbf{HoCW})$ and $\{X_n\}_{n\in\mathbb{N}}$ be an increasing sequence of subcomplexes such that $X = \bigcup_{n\in\mathbb{N}} X_n$ and let F be a contravariant functor as defined earlier. Then:

$$\{F(i_n)\}_{n\in\mathbb{N}} : F(X) \twoheadrightarrow \; colim_n \; F(X_n)$$

is a surjection.

Proof of Lemma 2.2.3:
We set $X_{-1} = \{x_0\}$. Consider the following sets

$$X' = \bigcup_{n \geqslant -1} [n-1, n]^+ \wedge X_n$$

$$A_1 = \bigcup_{k \geqslant -1, k \text{ odd}} [k-1,k]^+ \wedge X_k$$
$$A_2 = \bigcup_{k \geqslant -1, k \text{ even}} [k-1,k]^+ \wedge X_k$$

Then, we have that

- $A_1 \cup A_2 = X'$.
- $A_1 \cap A_2 = \bigcup_k \{k\}^+ \wedge X_k = \bigsqcup_k X_k \cong \bigvee_k X_k$. [5]
- $A_1 = \bigcup_{k \geqslant -1, k \text{ odd}} [k-1,k]^+ \wedge X_k \sim_{Hom} \bigcup_{k \geqslant -1, k \text{ odd}} \{k\} \wedge X_k = \bigsqcup_{k\ odd} X_k \cong \bigvee_{k\ odd} X_k$.
- $A_2 \sim_{Hom} \bigvee_{k\ even} X_k$.[6]

Thus, given any $\{x_n\} \in \operatorname{colim} F(X_n)$[7], using the wedge axiom, we get that $\exists y_1 \in F(A_1)$ with $y_1|_{X_k} = x_k$ for k odd. Similarly, $\exists y_2 \in F(A_2)$ with $y_2|_{X_k} = x_k$ for k even.
Then, let's consider $y_1|_{A_1 \cap A_2} = y_1|_{\bigvee_k X_k}$:

- for k odd, $y_1|_{X_k} = x_k$.
- for k even, using $i : X_k \hookrightarrow X_{k+1}$ and thus $i_{X_k} = i_{X_{k+1}} \circ i$,

$$y_1|_{X_k} = F(i_{X_k})(y_1) = F(i) \circ F(i_{X_{k+1}})(y_1) = F(i)(x_{k+1}) = x_k$$

It is the same proof for $y_2|_{A_1 \cap A_2}$ and thus

$$y_1|_{A_1 \cap A_2} = y_2|_{A_1 \cap A_2}$$

Using MV), $\exists y' \in F(X')$ with $y'|_{A_1} = y_1, y'|_{A_2} = y_2$. Thus, $y'|_{X_k} = x_k$. Now, using the fact that

$$X' = \bigcup_{n \geqslant -1} [n-1,n]^+ \wedge X_n \sim_{Hom} \bigcup_{n \geqslant -1} \{n\}^+ \wedge X_n \sim_{Hom} \bigcup_{n \geqslant -1} X_n = X$$

we get that $\exists y \in F(X)$ with $y|_{X_k} = x_k$ and thus $\{F(i_n)\}$ is surjective.

□ 2.2.3

Lemma 2.2.4.
For any map $f; X \to Y$ in **HoCW** *and F a contravariant functor as defined earlier. The following sequence*

$$F(X) \xleftarrow{F(f)} F(Y) \xleftarrow{F(j)} F(Y \cup_f CX)$$

is exact.

[5] Here, we are using the isomorphism $\varphi : \bigsqcup_k X_k \to \bigvee_k X_k$ with $\varphi(x)_i = x$ if $x \in X_i$, x_0 otherwise.
[6] Same way as for A_1
[7] In $\{x_n\}$, $x_n = F(i)(x_{n+1})$ with $i : X_k \hookrightarrow X_{k+1}$ the standard injection

Proof of Lemma 2.2.4:

- $\ker F(j) \subseteq$ Im F(f): Because $F(f) \circ F(j) = F(j \circ f) = F(\{*\}) = 0$[8].
- $\ker F(j) \supseteq$ Im F(f): Let $y \in F(Y)$ such that $F(f)(y) = 0$.

  Consider $A_1 = [0, \frac{1}{2}] \wedge X$, $A_2 = [\frac{1}{2}, 1]^+ \wedge X \cup_f Y$. Then $A_1 \cup A_2 = Y \cup_f CX$, $A_1 \cap A_2 = \{\frac{1}{2}\}^+ \wedge X$. Furthermore, we have that $A_2 \sim_{Hom} Y$and thus $F(A_2) = F(Y)$. Furthermore, $i : A_1 \cap A_2 \to Y$ is given by $id \wedge f$. Thus, let $y_1 = 0 \in F(A_1), y_2 = y \in F(A_2) = F(Y)$. Then $0 = y_1 = F(f)(y) = F(i)(y_2)$. Using MV), we get that $\exists z \in F(Y \cup_f CX)$ such that $F(j)(z) = z|_Y = F(f)(y)$.

□ 2.2.4

Now that we have some structure on F, we define the central notion of the Brown representation theorem.

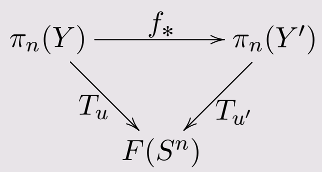
Definition 2.2.5 (Universal element)**.**
Let F be a contravariant functor as defined earlier.
*An element $u \in F(Y)$ is called **n-universal** if*

$$T_u : [S^q, Y] \to F(S^q)$$

*is an isomorphism for $q < n$ and an epimorphism (surjective morphism) for $q = n$. u is called **universal** if it is n-universal for all n.*
T_u is the natural transformation $T_u[f] = F(f)(u) \in F(S^q)$.

Any element of $F(Y)$ is at least a -1 universal element.

Remark 2.2.6.
If $f : Y \to Y'$ is a map in **CW** and $u \in F(Y), u' \in F(Y')$ are universal elements such that $F(f)(u') = u$, then f induces an isomorphism for all $n \in \mathbb{N}$ $f_* : \pi_n(Y) \cong \pi_n(Y')$. Indeed, the following diagram:



commute for all n. Because T_u and $T_{u'}$ are isomorphism, then so does f_*

[8]Using the fact that $j \circ f \sim_{Hom} \{*\}$

Now, let's show the core of Brown's theorem.

Lemma 2.2.7.
Let F be a contravariant functor as defined earlier.
For any $Y \in Ob(\mathbf{CW})$ and n-universal element $u_n \in F(Y)$, we can find a CW-complex Y' with $Y \subset Y'$ and a $n+1$-universal element $u_{n+1} \in F(Y')$ with $u_{n+1}|_Y = u_n$.

Proof of Lemma 2.2.7:
Let Y and u_n be a n-universal element in $F(Y)$. $\forall \lambda \in F(S^{n+1})$, we consider a copy of S^{n+1} named S^{n+1}_λ and we construct $Y \vee \bigvee_\lambda S^{n+1}_\lambda$.
Furthermore, if $n \geqslant 0$, $\forall \alpha \in \pi_n(Y)$ with $T_{u_n}(\alpha) = 0$, consider a representative $f_\alpha : S^n \to Y$ and attach a cell e^{n+1}_α to Y using $x = f(x) \in Y$, $\forall x \in \partial e^{n+1}_\alpha = S^n$. We define Y':

$$Y' = Y \bigcup_\alpha e^{n+1}_\alpha \vee \bigvee_\lambda S^{n+1}_\lambda$$

Now, consider the following map

$$g : \bigvee_\alpha S^n_\alpha \to Y$$

$$g \circ \iota_\alpha = f_\alpha$$

then, we get using $D^n \cong C(S^{n-1})$ that

$$Y' \cong (\bigvee_\lambda S^{n+1}_\lambda \vee Y) \cup_g C(\bigvee_\alpha S^n_\alpha)$$

It is thus the mapping cone of g. Thus, using 2.2.4, we get that

$$F(\bigvee_\alpha S^n_\alpha) \xleftarrow{F(g)} F(Y \vee \bigvee_\lambda S^{n+1}_\lambda) \xleftarrow{F(j)} F(Y')$$

is exact.
Furthermore, using W) on $F(Y \vee \bigvee_\lambda S^{n+1}_\lambda)$, we get that $\exists v$ such that $v|_Y = u_n$ and $v|_{S^{n+1}_\lambda} = \lambda$.
Then,

$$F(g)(v)|_{S^n_\alpha} = F(g)(u_n)|_{S^n_\alpha} = F(f_\alpha)(u_n) = T_{u_n}(\alpha) = 0$$

Thus, $v \in \ker(F(g)) = \operatorname{Im}(F(j))$. Thus, $\exists u_{n+1} \in F(Y')$ with $u_{n+1}|_{Y \vee \bigvee_\lambda S^{n+1}_\lambda} = v$.

Now, let's show that $u_{n+1} \in F(Y')$ is $n+1$-universal. Considering the following commutative diagram:

$$\begin{array}{ccc} \pi_q(Y) & \xrightarrow{i_*} & \pi_q(Y') \\ & {\scriptstyle T_{u_n}} \searrow \quad \swarrow {\scriptstyle T_{u_{n+1}}} & \\ & F(S^q) & \end{array}$$

- $\underline{q < n}$: Using the s.e.s

$$0 \to 0 = \pi_q(\bigvee_\alpha S^n_\alpha) \to \pi_q(Y) \xrightarrow{i_*} \pi_q\big(Y \cup_g C(\bigvee_\alpha S^n_\alpha)\big) \to 0$$

We get $i_* : \pi_q(Y) \cong \pi_q\big(Y \cup_g C(\bigvee_\alpha S^n_\alpha)\big)$ and with the wedge axiom on homotopy and $\pi_q(S^{n+1}) = 0$, we get that $i_* : \pi_q(Y) \cong \pi_q(Y')$. Thus, because our diagram commute and T_{u_n} is also an isomorphism, we get that $T_{u_{n+1}}$ is an isomorphism.

- $\underline{q = n}$: Similarly, to before, we have that

$$\pi_n(\bigvee_\alpha S^n_\alpha) \to \pi_n(Y) \xrightarrow{i_*} \pi_n\big(Y \cup_g C(\bigvee_\alpha S^n_\alpha)\big) \to 0$$

and using $\pi_n(S^{n+1}) = 0$, we have that $i_* : \pi_n(Y) \twoheadrightarrow \pi_n(Y')$ is an epimorphism. From our diagram, we had that $T_{u_{n+1}}$ was surjective.

Now, suppose $\exists \beta \in \pi_n(Y'), T_{u_{n+1}}(\beta) = 0$. Since i_* is surjective, we consider $\alpha \in \pi_n(Y), i_*(\alpha) = \beta$. Then, using

$$T_{u_n}(\alpha) = T_{u_{n+1}} \circ i_*(\alpha) = T_{u_{n+1}}(\beta) = 0$$

We can thus consider the cell $e^{n+1}_\alpha \in Y'$ and $f_\alpha : S^n \to e^{n+1}_\alpha$ a contractible space, which shows us that $i_*(\alpha) = 0$ and thus $T_{u_{n+1}}$ is injective.

- $\underline{q = n+1}$: $\forall \lambda \in F(S^{n+1})$,

$$T_{u_{n+1}}(i_\lambda) = F(i_\lambda)(u_{n+1}) = \lambda$$

with $i_\lambda : S^{n+1}_\lambda \to Y'$ the inclusion. Thus, $T_{u_{n+1}}$ is an epimorphism for $q = n + 1$.

□ 2.2.7

The later lemma induce the following corollary

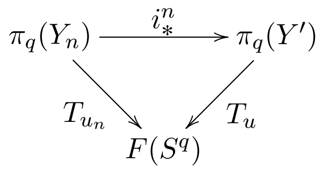
Corollary 2.2.8.
Let F be a contravariant functor as defined earlier.
$\forall Y \in Ob(\mathbf{CW})$ and $\forall v \in F(Y)$, we can find a CW complex Y' containing Y and a universal element $u \in F(Y')$ with $u|_Y = v$.

Proof of Corollary 2.2.8:
We take $Y_{-1} = Y, u_{-1} = v$. Using inductively the previous lemma 2.2.7, we construct a sequence $\{Y_n\}$ and $\{u_n\}$ s.t. $Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots$ and $u_n|_{Y_{n-1}} = u_{n-1}$.
Then, consider

$$Y' = \bigcup_{n \geqslant -1} Y_n$$

Using lemma 2.2.3, because $\{u_n\} \in \text{colim } F(Y_n)$, we consider $u \in Y'$ s.t. $\{i_n\}(u) = \{u_n\}$.

Now, let's show that Y' and u are universal. We consider $q < n$ and the following diagram:



Because $Y' \backslash Y_n$ is composed of cells of dimension greater that n [9], we have that, similarly to the proof of the lemma 2.2.7, i_*^n is an isomorphism for $q < n$. Thus, so does T_u. Because we can choose n freely, we get that u is universal.

□ 2.2.8

Now that we have a notion of universality on sphere, S^q, we want to extend that on all CW complexes, like for 1.4.24. To do so, consider the following lemma.

Lemma 2.2.9.
Let F be a contravariant functor as defined earlier and Y be a space with universal element $u \in F(Y)$, $X, A \in Ob(\mathbf{CW})$ with $A \subset X$. Let $g \in \mathbf{CW}(A, Y)$ and $v \in F(X)$ such that $v|_A = F(g)(u)$. Then, $\exists h \in \mathbf{CW}(X, Y)$ such that $h|_A = g$ and $v = F(h)(u)$.

Proof of Lemma 2.2.9:
we define the following sets

$$T = \Big((I^+ \wedge A) \vee X \vee Y\Big)/\sim \text{ with } (0, a) \sim a \in X \text{ and } (1, a) \sim g(a) \in Y$$

$$A_1 = ([0, 1/2]^+ \wedge A) \cup X, A_2 = Y \cup_g ([1/2, 1]^+ \wedge A)$$

Then $A_1 \cup A_2 = T, A_1 \cap A_2 = \{1/2\}^+ \wedge A \cong A$. Using $[0, 1/2] \sim_{Hom} \{0\}$, we get $f_1; A_1 \to X$[10] an homotopy equivalence and same with $f_2 : A_2 \to Y$[11].
We thus have that $\exists \overline{v} \in F(A_1)$ with $\overline{v}|_X = v$ and $\exists \overline{u} \in F(A_2)$ with $\overline{u}|_Y = u$. Furthermore

$$\overline{v}|_{A_1 \cap A_2} = F(f_1)(v|_A) = F(f_1) \circ F(g)(u) = F(f_2)(u) = \overline{u}|_{A_1 \cap A_2}$$

Thus, using MV), we get that $\exists w \in F(T)$ with $w|_X = v, w|_Y = u$.
Now, using corollary 2.2.8, we can extend T into a CW-complex Y' with an universal element $u' \in F(Y')$ with $u'|_T = w$. Using $j : Y \hookrightarrow Y'$ the inclusion, we get that

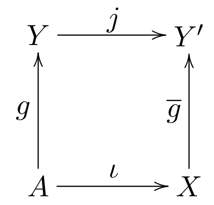
$$F(j)(u') = u'|_Y = w|_Y = u.$$

Thus, using remark 2.2.6, we get that $\forall n \in \mathbb{N}$

$$j_* : \pi_n(Y) \cong \pi_n(Y').$$

[9] by construction, see lemma 2.2.7
[10] $f_1(t, a) = a, f|_X = id$
[11] $f_2(t, a) = g(a), f|_Y = id$

Now, let's consider the inclusion $i : A \wedge I^+ \subset T \hookrightarrow Y'$. we can see that it is in fact an homotopy between $i_X : X \hookrightarrow Y'$ restricted to A and $j \circ g$. That is $i_X|_A \sim_{Hom} j \circ g$. Using the homotopy extension property 1.4.16, we get that $\exists \overline{g} : X \to Y'$ such that $j \circ g = \overline{g}|_A$ and $i_X \cong \overline{g}$.
We thus have the following commutative diagram:



with j a weak homotopy equivalence. Then, using Whitehead theorem 1.4.17, we get that j is an homotopy equivalence with homotopy inverse $\widetilde{j}$. Then, we define $h = \widetilde{j} \circ \overline{g}$ and by construction, $j \circ h \sim_{Hom} \overline{g} \sim_{Hom} i_X$. Thus:

$$F(h)(u) = F(j \circ h)(u') = F(i_X)(u') = v$$

□ 2.2.9

Theorem 2.2.10 (Brown's representation theorem)**.**
If $F : \mathbf{HoCW} \to \mathbf{Gp}$ is a contravariant functor satisfying W) and MV), then there is a classifying space $Y \in Ob(\mathbf{CW})$ and an universal element $u \in F(Y)$ such that

$$T_u : [-, Y] \cong F$$

is a natural equivalence.

Proof of Theorem 2.2.10:
Let $\{*\}$ be the base point. Using corollary 2.2.8, we get that $\exists Y \in Ob(\mathbf{CW})$ with an universal element $u \in F(Y)$. Now, let's show that

$$T_u : [X, Y] \to F(X)$$

is bijective $\forall X \in \mathbf{CW}$.

- *surjective:* Let $v \in F(X)$. We consider $A = \{*\}$ and $g : A \to Y$ the base point function. By 2.2.2, $v|_A = 0 = F(g)(u)$ Then, using lemma 2.2.9, we get that $\exists h \in \mathbf{CW}(X, Y)$ such that $h(*) = *$ and $T_u(h) = F(h)(u) = v$.

  Thus T_u is surjective.

- *injective:* Let $T_u[g_0] = T_u[g_1]$ for two maps $g_0, g_1 : X \to Y$. Let $X' = X \wedge I^+$, $A' = X \wedge \{0,1\}^+$ and
$$g : A' \to Y$$

$$g(x,0) = g_0(x),\ \ g(x,1) = g_1(x).$$

We also define

$$p : X' \to X$$

$$p(x,t) = x$$

and let's consider $v = F(g_0) \circ F(p)(u) \in F(X')$. Then

$$v|_{X \wedge \{0\}^+} = F(g_0)(u) = F(g)(u)|_{X \wedge \{0\}^+}$$

$$v|_{X \wedge \{1\}^+} = F(g_0)(u) = T_u(g_0) = T_u(g_1) = F(g_1)(u) = F(g)(u)|_{X \wedge \{1\}^+}$$

This means that $v|_{A'} = F(g)(u)$

Then, using lemma 2.2.9, we get $h \in \mathbf{CW}(X', Y)$ such that $h|_{A'} = g$. But then h is a homotopy between g_0 and g_1. Thus, T_u is injective.

To show the naturality part, consider $f \in \mathbf{HoCW}(X, Z)$ and the following diagram:

$$\begin{array}{ccc} [X,Y] & \xrightarrow{T_u(X)} & F(X) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle F(f)} \\ [Z,Y] & \xrightarrow{T_u(Z)} & F(Z) \end{array}$$

This diagram commute: $T_u(f^*(\mu)) = T_u(\mu \circ f) = F(\mu \circ f)(u) = F(f) \circ F(\mu)(u) = F(f) \circ T_u(\mu)$

□ 2.2.10

Now, that we know that we can represent covariant functors F that follows W) and MV), can we represent natural transformation between them? The answer is yes.

Theorem 2.2.11.
Let $F, F' : \mathbf{HoCW} \to \mathbf{Gp}$ be contravariant functors with classifying space Y, Y' and universal element $u \in F(Y), u' \in F(Y')$.
If $T : F \to F'$ is a natural transformation, then $\exists ! [f] \in [Y, Y']$ such that the following diagram commute:

$$\begin{array}{ccc} [X,Y] & \xrightarrow{f_*} & [X,Y'] \\ {\scriptstyle T_u(X)}\Downarrow & & \Downarrow{\scriptstyle T_{u'}(X)} \\ F(X) & \xrightarrow{T(X)} & F'(X) \end{array}$$

Furthermore, if T is a natural equivalence, then we have that f is an homotopy equivalence.

Proof of Theorem 2.2.11:
Consider $[f] \in [Y, Y']$ such that

$$T(Y) \circ T_u(Y)(id_Y) = T_{u'}(Y)([f])$$

Then, $T_{u'}(Y)([f]) = F'(f)(u') = T(id_F(Y)) = id_{F'(Y)}$ and thus

$$T'_u(X) \circ f_*(\mu) = F'(f \circ \mu)(u') = F'(\mu) \circ F'(f)(u') = F'(\mu) = T \circ T_u(X)(\mu)$$

For the unicity part, suppose g_* is such that the diagram commute. Then,

$$T'_u(Y)(g) = T'_u(Y) \circ g_*(id_Y) = T \circ T_u(Y)(id_Y)$$

That is, $[g] = [f]$

If T is a natural equivalence, then using the diagram, we have that $f_* : \pi_n(Y) \cong \pi_n(Y')$ and thus, using Whitehead spectral theorem 1.4.17, we get that f is an homotopy equivalence.

□ 2.2.11

Remark 2.2.12.
If we have proven Brown's representation theorem for contravariant functors into **Gp**, we can see that our proofs don't require any group structure. So Brown's representation theorem works also (and was historically first proven) for

$$F : \mathbf{HoCW} \to \mathbf{Set}$$

2.2.2 Consequences on spectra

Now that we have proven Brown's representation theorem, we want to see what consequences this gives to spectra. To do so, we need to define a subcategory of **Sp**.

Definition 2.2.13 (Ω-spectra)**.**
We define an Ω-spectrum as a spectrum E such that duals of p_n using

$$A : [\Sigma E_n, E_{n+1}] \cong [E_n, \Omega E_{n+1}]$$

named $p'_n : E_n \hookrightarrow \Omega E_{n+1}$[12] *are weak homotopy equivalences.*
In fact, we define the subcategory $\mathbf{\Omega Sp} \subset \mathbf{Sp}$:

$$Ob(\mathbf{\Omega Sp}) = \left\{ E \in \mathbf{Sp} |\ E \textit{ is an } \Omega\textit{-spectra} \right\}$$

$$\mathbf{\Omega Sp}(E, F) = \mathbf{Sp}(E, F)$$

and the subcategory $\mathbf{Ho\Omega Sp} \subset \mathbf{HoSp}$:

$$Ob(\mathbf{Ho\Omega Sp}) = Ob(\mathbf{\Omega Sp})$$

$$\mathbf{Ho\Omega Sp}(E, F) = [E, F]$$

One very important property of Ω-spectra is that the expression of their induce cohomology is greatly simplified.

Theorem 2.2.14.
If $E \in \mathbf{\Omega Sp}$, then $\forall n \in \mathbb{Z}$, we have a natural isomorphism

$$T^n(-) : E^n(-) \cong [-, E_n].$$

Proof of Theorem 2.2.14:
Let $k^n(X) = [X, E_n]$. We consider the following natural equivalence chain

$$[\Sigma X, E_{n+1}] \xrightarrow[\cong]{A} [X, \Omega E_{n+1}] \xleftarrow[\cong]{p'_{n*}} [X, E_n]$$

This induce $\overline{\sigma^n} : k^{n+1}(\Sigma -) \cong k^n(-)$. Furthermore, $\forall A \subset X$, the sequence

$$[A, E_n] \xleftarrow{\iota^*} [X, E_n] \xleftarrow{j^*} [X \cup CA, E_n]$$

[12] $[p'_n] = A[p_n]$

is exact from 2.2.1 and 2.2.4.
Thus, k^* is a cohomology theory on **HoCW**. Since for an collection $\{X_\alpha\}_{\alpha\in A}$ is such that

$$\{i_\alpha^*\} : \big[\bigvee_\alpha X_\alpha, E_n\big] \cong \prod_\alpha [X_\alpha, E_n]$$

using 2.2.1, k^* follows the wedge axiom.

Now, let's construct our $T^n : k^n \to E^n$. Let $f \in \mathbf{CW}(X, E_n)$. We define the spectrum $\overline{\Sigma^\infty} X$ with

$$(\overline{\Sigma^\infty} X)_k = \begin{cases} * & k < n \\ \Sigma^{k-n} X & k \geqslant n \end{cases}$$

This is a cofinal subspectrum of $\Sigma^{-n}\Sigma^\infty X$ and we use it to define a map

$$\overline{f} : \Sigma^{-n}\Sigma^\infty X \to E$$

$$\overline{f} = [\overline{\Sigma^\infty} X, f'] \text{ with } f' = (\cdots, *, f, \Sigma f, \Sigma^2 f, \cdots)$$

$$T^n([f]) = [\Sigma^n \overline{f}] \in [\Sigma^\infty X, \Sigma^n E] = E^n(X)$$

We see that T^n is well defined. indeed, if $f \sim_{Hom} g$ using $h : X \wedge I^+ \to E_n$, then maps

$$\overline{h} : \Sigma^{-n}\Sigma^\infty X \wedge I^+ \to E$$

$$\overline{h} = [\overline{\Sigma^\infty} X \wedge I^+, h'], h' = (h, \Sigma h, \cdots)$$

is an homotopy between $\overline{f}$ and $\overline{g}$.

Furthermore, by considering the following diagram, we see that it commute.

$$\begin{array}{ccc}
[X, E_n] & \overset{\overline{\sigma^n}}{\Longleftarrow} & [\Sigma X, E_{n+1}] \\
{\scriptstyle T(X)}\downarrow & & \downarrow{\scriptstyle T^{n+1}(\Sigma X)} \\
[\Sigma^\infty X, \Sigma^n E] & \overset{\sigma^n}{\Longleftarrow} & [\Sigma^\infty \Sigma X, \Sigma^{n+1} E]
\end{array}$$

Insuring that T^* is a natural transformation.
Now, for the isomorphism part, using the following commutative diagram:

$$\begin{array}{ccc}
\pi_k(E_{n+k}) & \xrightarrow{\Sigma} & \pi_{k+1}(\Sigma E_{n+k}) \\
\Big\| {\scriptstyle (p'_{n+k})_*} & & \downarrow{\scriptstyle (p_{n+k})_*} \\
\pi_k(\Omega E_{n+k+1}) & \overset{A}{=\!=\!=} & \pi_{k+1}(E_{n+k+1})
\end{array}$$

we get the isomorphism

$$k^n(S^0) = \pi_0(E_n) \cong \ \text{colim}_k\ \pi_k(E_{n+k}) = \ \text{colim}_{k'}\ \pi_{k'-n}(E_{k'}) \cong \pi_{-n}(E) = E^n(S^0).$$

Meaning that for all $n \in \mathbb{Z}$, $T^n(S^0) : k^n(S^0) \cong E^n(S^0)$.
Thus, by using theorem 1.4.24, we get that T^* is a natural equivalence.

□ 2.2.14

Similarely to 2.1.7, we can lower the requirement to construct an Ω spectrum.

Proposition 2.2.15.
Let $\{E_n, \widetilde{p_n}\}_{n\in\mathbb{Z}}$ be a sequence of CW complexes with $\widetilde{p_n} : E_n \to \Omega E_{n+1}$ weak homotopy equivalence. Then, $\exists E' \in Ob(\mathbf{Sp})$ an Ω-spectrum such that there is a natural isomorphism.

$$E'^n(-) \cong [-, E_n]$$

Proof of Proposition 2.2.15:
Using $A : [\Sigma E_n, E_{n+1}] \cong [E_n, \Omega E_{n+1}]$, we consider $p_n : \Sigma E_n \to E_{n+1}$ such that $A([p_n]) = [\widetilde{p_n}]$. Then, using lemma 2.1.7, we get $E' \in \mathbf{Sp}$ such that $r_n : E'_n \sim_{Hom} E_n$. Now, let's show that E' is an Ω-spectrum. For this, take $\widetilde{p'_n} \in A[\iota]$. The following commutative diagram gives us our wanted result:

$$\begin{array}{ccccc}
\pi_k(E'_n) & \xrightarrow{\widetilde{p'_n}} & \pi_k(\Omega E'_{n+1}) & \overset{A}{\Longleftarrow} & \pi_{k+1}(E'_{n+1}) \\
\Big\Downarrow (r_n)_* & & & & \Big\Downarrow (r_{n+1})_* \\
\pi_k(E_n) & \overset{\widetilde{p_n}}{\Longrightarrow} & \pi_k(\Omega E_{n+1}) & \overset{A}{\Longleftarrow} & \pi_{k+1}(E_{n+1})
\end{array}$$

Thus, E' is an Ω-spectrum. Now, using theorem 2.2.14, we get that

$$E'^n(-) \cong [-, E'_n] \cong [-, E_n]$$

□ 2.2.15

We can now prove an extension of Brown's representation theorem for cohomology.

Theorem 2.2.16 (Brown's theorem on cohomology)**.**
Let $k^* : \mathbf{HoCW} \to \mathbf{Ab}$ be a reduced cohomology that follows the wedge axiom. Then, $\exists E \in \mathbf{\Omega Sp}$ and a natural equivalence

$$T : E^*(-) \cong k^*(-)$$

Proof of Theorem 2.2.16:
Let $F = k^n(-) : \mathbf{HoCW} \to \mathbf{Ab}$. By 1.3.8, we have that F follows the MV). Then, by using Brown's representation theorem 2.2.10, we get that $\exists E_n \in Ob(\mathbf{CW})$ and a natural transformation T_n such that

$$T_n : [-, E_n] \cong F$$

Now, we have natural equivalences

$$[X, \Omega E_{n+1}] \xrightarrow[\cong]{A^{-1}} [\Sigma X, E_{n+1}] \xrightarrow[\cong]{T_{n+1}} k^{n+1}(\Sigma X) \xrightarrow[\cong]{\sigma} k^n(X) \xrightarrow[\cong]{T_n^{-1}} [X, E_n]$$

Then $\phi_n : E_n \to \Omega E_{n+1}$ with $[\phi_n] = A \circ T_{n+1}^{-1} \circ \sigma^{-1} \circ T_n([id_{E_n}])$ and $\psi_n : \Omega E_{n+1} \to E_n$ with $[\psi_n] = T_n^{-1} \circ \sigma \circ T_{n+1} \circ A^{-1}([id_{\Omega E_{n+1}}])$ are such that:

$$[\phi_n \circ \psi_n] = A \circ T_{n+1}^{-1} \circ \sigma^{-1} \circ T_n[\psi_n] = A \circ T_{n+1}^{-1} \circ \sigma^{-1} \circ T_n \circ T_n^{-1} \circ \sigma \circ T_{n+1} \circ A^{-1}[id_{\Omega E_{n+1}}] = [id_{\Omega E_{n+1}}]$$

$$[\psi_n \circ \phi_n] = T_n^{-1} \circ \sigma \circ T_{n+1} \circ A^{-1}[\phi_n] = T_n^{-1} \circ \sigma \circ T_{n+1} \circ A^{-1} A \circ T_{n+1}^{-1} \circ \sigma^{-1} \circ T_n[id_{E_n}] = [id_{E_n}].$$

Therefore, they are weak homotopy equivalences.
Thus, $\{E_n, \phi_n\}$ induce by proposition 2.2.15 that $\exists E'$ an Ω-spectrum such that we have a natural equivalence

$$E'^*(-) \cong [-, E_*] \cong k^*(-)$$

□ 2.2.16

Now, to go further in this subject, we need the following lemma used to ease the following constructions.

Lemma 2.2.17.
Let $E, F \in Ob(\mathbf{Sp})$ and let $\{f_n : E_n \to F_n\}_{n \in \mathbb{N}}$. If $E_n = *$ for $n < N$ and if $\Sigma' f_n \sim_{Hom} f_{n+1}|_{\Sigma' E_n}$. Then,

$$\exists \tilde{f} : E \to F$$

$$\tilde{f}_n \sim_{Hom} f_n$$

Furthermore, if $\forall n \geqslant N$, $\lim^1 F^{n-1}(E_n) = 0$ as defined here, then $\tilde{f}$ is unique up to homotopy. (i.e. if $g : E \to F$ with $g_n \sim_{Hom} \tilde{f}_n$, then $g \sim_{Hom} \tilde{f}$).

Proof of Lemma 2.2.17:
Take $\tilde{f}_n = f_n = *$ for $n < N$. Then, by iteration, we have that $\Sigma' f_n \sim_{Hom} f_{n+1}|_{\Sigma' E_n}$. Thus, using the homotopy extention on CW complex 1.4.16, we can find $\tilde{f}_{n+1}$ such that $f_{n+1} \sim_{Hom} \tilde{f}_{n+1}$ and $\tilde{f}_{n+1}|_{\Sigma E_n} = \Sigma \tilde{f}_n$.

Thus, the collection of $\{\tilde{f}_n\}$ gives us a spectral map.
Now, let $\{E^n\}_{n \in \mathbb{N}}$ be an increasing sequence of subspectra (a fibration) of E such that $\cup_{n \in \mathbb{N}} E^n = E$.

Using 1.4.21, we have $\forall m, q \in \mathbb{N}$ the s.e.s.

$$0 \to \lim_n^1 F^{q-1}(E_m^n) \to F^q(E_m) \to \ \lim_n F^q(E_m^n) \to 0$$

With 2.1.47 and the fact that $E_n = *$ for $n < N$, we have that $F^q(E) = \lim_m F^{q+N}(E_{m+M})$. Thus, because we have the s.e.s.

$$0 \to \lim_n^1 F^{q+N-1}(E_{m+N}^n) \to F^{q+N}(E_{m+N}) \to \ \lim_n F^{q+N}(E_{m+N}^n) \to 0$$

going to limit on m and using permutation of limit, we get

$$\begin{array}{ccccccccc} & & \lim_n^1 F^{q-1}(E^n) & & & & \lim_n F^q(E^n) & & \\ & & \| & & & & \| & & \\ 0 & \longrightarrow & \lim_m \lim_n^1 F^{q+N-1}(E_{m+N}^n) & \longrightarrow & F^q(E) & \longrightarrow & \lim_m \lim_n F^{q+N}(E_{m+N}^n) & \longrightarrow & 0 \end{array}$$

A more thorough look at the inner working of the proof, and details on this limit permutation can be found in the proof of 1.4.16.
Going along, we get the s.e.s.

$$0 \to \lim_n^1 F^{q-1}(E^n) \to F^q(E) \to \ \lim_n F^q(E^n) \to 0$$

Then, consider the fibration E^n defined as

$$E_m^n = \begin{cases} E_m & m \leqslant n \\ \Sigma^{m-n} E_n & m \geqslant n \end{cases}$$

We have, using 2.1.47, the s.e.s.

$$0 \to \lim_n^1 F^{-1}(E^n) \to [E, F] \to \ \lim_n F^0(E^n) \to 0.$$

Then, we see that $\Sigma^{-n}\Sigma^\infty E_n$ is cofinal in E^n. Thus,

$$F^q(E^n) \cong F^q(\Sigma^{-n}\Sigma^\infty E_n) \cong F^{n+q}(\Sigma^\infty E_n) = F^{n+q}(E_n).$$

Thus, because $\lim_n^1 F^{n-1}(E_n) = 0$, we get that

$$[E, F] \cong \ \lim_n F^n(E_n)$$

$$[f] \cong \ \lim_n [f_n]$$

proving the unicity up to homotopy of $\tilde{f}$.

□ 2.2.17

We can now look at an equivalent to 2.2.11 for cohomology.

Theorem 2.2.18.
If $E, E' \in \mathbf{\Omega Sp}$ and $T : E^*(-) \to E'^*(-)$ is a natural transformation of cohomology theories on **HoCW**, *then there is a spectral map $f : E \to E'$ such that $T_n = (\Sigma^n f)_*$*
In fact, if T is a natural equivalence, then f is an homotopy equivalence.
Furthermore, if $\lim^1_n E'^{n-1}(E_n) = 0$[13] with $\lim^1$ as defined here, then f is unique up to homotopy.

Proof of Theorem 2.2.18:
Consider $T_n : E^n(-) \to E'^n(-)$. Using theorem 2.2.11, we get the unique $[f_n] \in [E_n, E'_n]$ such that $T_n = (f_n)_*$. Now, we have to show that $\Sigma f_n = f_{n+1}|_{\Sigma E_n}$.
By considering the following commutative diagram:

$$\begin{array}{ccc}
[E_n, E'_n] & \xrightarrow{\quad\Sigma\quad} & [\Sigma E_n, \Sigma E'_n] \\
\Big\| (p'_n)_* & & \Big\downarrow \iota_* \\
[E_n, \Omega E'_{n+1}] & \overset{A^{-1}}{=\!=\!=\!=} & [\Sigma E_n, E'_{n+1}]
\end{array}$$

We get that that $\exists g_n : E_n \to E'_n$ such that $[\Sigma g_n] = [f_{n+1}|_{\Sigma E_n}]$. Then, using the following commutative diagram

$$\begin{array}{ccccccccc}
[X, E_n] & \Longrightarrow & E^n(X) & \overset{\sigma^{-1}}{\Longleftarrow} & E^{n+1}(\Sigma X) & \Longleftarrow & [\Sigma X, E^{n+1}] & \overset{\iota_* \circ \Sigma}{\Longleftarrow} & [X, E_n] \\
(f_n)_* \Big\downarrow & & \Big\downarrow T_n(X) & & T_{n+1}(\Sigma X) \Big\downarrow & & (f_{n+1})_* \Big\downarrow & & \Big\downarrow (g_n)_* \\
[X, E'_n] & \Longrightarrow & E'^n(X) & \overset{\sigma'^{-1}}{\Longleftarrow} & E'^{n+1}(\Sigma X) & \Longleftarrow & [\Sigma X, E'^{n+1}] & \overset{\iota_* \circ \Sigma}{\Longleftarrow} & [X, E'_n]
\end{array}$$

gives us by unicity of f_n up to homotopy that $[f_n] = [g_n]$, which means that $[\Sigma f_n] = [\Sigma g_n] = [f_{n+1}|_{\Sigma E_n}]$. i.e.

$$\Sigma f_n \sim_{Hom} f_{n+1}|_{\Sigma E_n}$$

Now, using the cofinal subspectrum $E^{\geqslant}$, where $E^{\geqslant}_n = E_n$ for $n \geqslant 0$, $E^{\geqslant}_n = *$ otherwise, we can use 2.2.17, We thus get a spectral map $\overline{f} : E \to E'$ (with $\overline{f}$ defined on $E^{\geqslant}$) such that $\overline{f}_n \sim_{Hom} f_n$[14]. Then:

[13] Or simply, $\text{colim}_n[E_n, E'_n] \cong [E, E']$.
[14] If it is apparent why it work for $n \geqslant 0$, we can extend this to negative by induction by adding g_n bellow $\bar{f}_0$, we know that $g_n \sim_{Hom} f_n$. At the end, for any n, we can find a suitable representative.

$$
\begin{array}{ccc}
[\Sigma^\infty(-), \Sigma^n E] & \xrightarrow{\Sigma^n \overline{f}_*} & [\Sigma^\infty(-), \Sigma^n E'] \\
\| & & \| \\
[-, E_n] & \xrightarrow{(f_n)_*} & [-, E'_n] \\
\| & & \| \\
E^n(-) & \xrightarrow{T} & E'^n(-)
\end{array}
$$

gives us that $\overline{f}$ represent the natural transformation T.
If T is a natural equivalence, then using that $\pi_{-n}(E) = [\Sigma^{-n}\mathbb{S}, E] \cong [\mathbb{S}, \Sigma^n E] = E^n(S^0)$ and the previous diagram, we get that

$$\bar{f}_* : \pi_{-n}(E) \cong \pi_{-n}(E')$$

and thus, using Whitehead spectral theorem 2.1.32, that f is an homotopy equivalence.

To show the unicity, using 2.2.17, we get that

$$\mathrm{colim}_n[E_n, E'_n] \cong [E^{\geqslant}, E'] \cong [E, E']$$

We have by 2.2.11 that $[\overline{f_n}]$ are unique up to homotopy and therefore so is $\overline{f}$.

□ 2.2.18

Note that, contrary to Brown's representation theorem, we don't always get unicity of our map up to homotopy. Let's now define the following category of reduced cohomology, called stable.

Definition 2.2.19 (Stable cohomology category)**.**
We define the stable cohomology category $\mathbf{cohom}_S$*:*

$$Ob(\mathbf{cohom}_S) = \Big\{ h^*(-) |\ h^*(-) \textit{ is a cohomology that follows the wedge and WHE axiom} \Big\}$$

$$\mathbf{cohom}_S\big(h^*(-), k^*(-)\big) = \Big\{ T : h^*(-) \to k^*(-) |\ T \textit{ is a natural transformation} \Big\}$$

Then, using the previous two theorem, we get the following theorem.

Corollary 2.2.20.
Using the following functors

$$V : \mathbf{Ho\Omega Sp} \to \mathbf{cohom}_S$$
$$E \to E^*(-)$$
$$f \to (\Sigma^n f)_*$$

$$W : \mathbf{cohom}_S \to \mathbf{Ho\Omega Sp}$$
$$h^*(-) \to E \textit{ given by 2.2.16}$$
$$T \to f \textit{ given by 2.2.18}$$

We get an category equivalence

$$\mathbf{Ho\Omega Sp} \cong \mathbf{cohom}_S$$

Although a very nice and powerful result, we are not fully satisfied with it because we are dealing with $\mathbf{\Omega Sp}$ and not $\mathbf{Sp}$.
To extend our result, we need this topological result.

Lemma 2.2.21.
Let $X \in \mathbf{Top}_\bullet$ be compact, $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of T_1 pointed topological spaces such that $B_i \hookrightarrow B_{i+1}$ is closed. We can define colim $B_i = B \in \mathbf{Top}_\bullet$. Then

$$colim_{i\in\mathbb{N}}\ Hom(X, B_i) \cong Hom(X, B)$$

$$\{f_i\}_{i\in\mathbb{N}} \to \overline{f} \textit{ with } \overline{f}(x) = colim\{f_i\}(x)$$

Proof of Lemma 2.2.21:
wlog, we may suppose that $B = \bigcup_{i\in\mathbb{N}} B_i$ and that B has the topology of the union. Then, the B_i are closed and A is closed $\iff \forall i, A \cap B_i$ is closed in B_i

If the injection is trivial, we still have to show the surjection. Let f be such that it doesn't factors through. Now, let $d_0 \in B$. Then, $\exists i_0$ s.t. $d_0 \in B_{i_0}$. Then, consider $f^{-1}(B_{i_0})$ a closed subset of A. We consider $k_0 \in X\backslash f^{-1}(B_{i_0})$ and $d_1 = f(k_0)$. $\exists i_1 > i_0$ such that $d_1 \in B_{i_1}, \cdots$

We thus have an increasing sequence of i_n with $f(k_n) \in B_{i_{n+1}}$. Then, we define $S_n = \{f(k_{n+i})\}_{i\in\mathbb{N}} \subset f(X)$. $S_n \cap B_m$ is a finite union of points. It is thus closed.

We see that finite intersection of $\bigcap_{i=0}^{m} S_{n_i} \neq \emptyset$ but $\bigcap_{n\in\mathbb{N}} S_n = \emptyset$, which contradict the fact that $f(X)$ is compact because image of a compact.

□ 2.2.21

Corollary 2.2.22.
Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of CW-complexes $B_i \hookrightarrow B_{i+1}$ (with cellular maps). Then

$$colim_{i\in\mathbb{N}}\ \Omega(B_i) \cong \Omega(colim_{i\in\mathbb{N}}\ B_i)$$

Proof of Corollary 2.2.22:
this comes from the fact that S^1 is compact, that subcomplexes of CW-complexes are closed and that they are Hausdorff.

□ 2.2.22

Then previous corollary allows us to demonstrate the following theorem.

Theorem 2.2.23.
*Let $E \in$ **Sp**. Then we can find $\omega E \in$ **ΩSp** and $r : E \to \omega E$ an homotopy equivalence.*

Proof of Theorem 2.2.23:
Let $\widetilde{p_n} : E_n \to \Omega E_{n+1}$ be the dual of $p_n : E_n \hookrightarrow E_{n+1}$ by A. Then we define

$$E'_n = \text{colim}_k \Omega^k E_{n+k}$$

using $\Omega^k(p_{n+k}) : \Omega^k E_{n+k} \hookrightarrow \Omega^{k+1} E_{n+k+1}$ [15]. Thanks to the previous corollary 2.2.22, we get that

$$\begin{array}{rcl} \Omega E'_{n+1} & = & \Omega(\text{colim}_k \Omega^k E_{n+1+k}) \\ & \cong & \text{colim}_k\ \Omega(\Omega^k E_{n+1+k}) \\ & = & \text{colim}_k\ \Omega^{k+1} E_{n+1+k} \\ & = & \text{colim}_k\ \Omega^{k+1} E_{n+k+1} \\ & = & \text{colim}_{k'}\ \Omega^{k'} E_{n+k'} \\ & = & E'_n \end{array}$$

Thus, using this homeomorphism p'_n, we can use proposition 2.2.15 on $\{E'_n, p'_n\}$ that give us $\omega E \in \mathbf{\Omega Sp}$ such that $\omega E^*(-) \cong [-, E'_n]$.

We know that

$$(\omega E)_n = E'_n \wedge \{n\}^+ \cup \bigsqcup_{k<n} \Sigma^{n-k} E'_k \wedge [k, k+1]^+_{/\sim}.$$

[15] It is indeed injective. Using the nature of $A(f) = \hat{f}$, $\hat{f}(x) = \hat{f}(y) \iff f(x,z) = f(y,z) \Rightarrow x = y$.

Then, by using $\iota_n : E_n \hookrightarrow \text{colim}_k \Omega^k E_{n+k} = E'_n$ the standard injection, we can construct

$$r_n : E_n \to (\omega E)_n$$

$$r_n(x) = \iota_n(x) \wedge \{n\} \cup \bigsqcup_{k<n} \Sigma^{n-k} \iota_k(x) \wedge \{k\}$$

Then, we have that

$$\begin{array}{rcl}
\Sigma r_n(x) & = & \Sigma\Big(\iota_n(x) \wedge \{n\} \cup \bigsqcup_{k<n} \Sigma^{n-k} \iota_k(x) \wedge \{k\}\Big) \\
 & = & \Sigma \iota_n(x) \wedge \{n\} \cup \bigsqcup_{k<n} \Sigma^{n+1-k} \iota_k(x) \wedge \{k\} \\
 & = & r_{n+1}|_{\Sigma(\omega E)_n}
\end{array}$$

Thus, we have a spectral function $r : E \to \omega E$
Now, to prove that r is an homotopy equivalence, we will consider the following diagram:

$$\begin{array}{ccc}
\pi_n(E) & \xrightarrow{\quad r_* \quad} & \pi_n(\omega E) \\
\| & & \| \; T \\
\| & & \pi_0(E'_{-n}) \\
\| & & \| \\
\| \; \alpha & & \pi_0(\text{ colim}_k \Omega^k E_{-n+k}) \\
\| & & \| \; U \\
\| & & \text{colim}_k \pi_0(\Omega^k E_{-n+k}) \\
\| & & A \; \| \\
\text{colim}_{k'} \pi_{n+k'}(E'_k) & = & \text{colim}_k \pi_k(E_{-n+k})
\end{array}$$

with T given by 2.2.15, U given by 2.2.21 applied at the homotopy category.
Thus, we have that r is a weak homotopy equivalence and thus an homotopy equivalence by using Whitehead spectral theorem 2.1.32.

□ 2.2.23

In fact, the construction of ωE can be made into a functor, named spectrification [16]. We thus get the following result.

[16] details can be found in [GJMS+06], page 3-10

Corollary 2.2.24.
Using spectrification $\omega : \mathbf{HoSp} \to \mathbf{Ho\Omega Sp}$ *and the forgetful functor* $\mathcal{U} : \mathbf{Ho\Omega Sp} \to \mathbf{HoSp}$*, we get a category equivalence*

$$\mathbf{HoSp} \cong \mathbf{Ho\Omega Sp}.$$

Combined with 2.2.20, this gives us that

$$\mathbf{HoSp} \cong \mathbf{cohom}_{S}.$$

In fact, there exists a similar construction to show that homology on **HoCW** that follow the wedge axioms and **HoSp** are equivalent categories. The proof is rather different from the cohomology one, and quite interesting. It can be seen in [Swi17] 14.35.

Example 2.2.25.
Here are a view examples of representative spectra over cohomology.

- *We can say that the singular reduced cohomology modulo* A $\left(\tilde{H}^*(-, A)\right)$ *is induced by the* ***Eilenberg-Mac line spectrum*** *noted* HA*. More information on this family of spectrum can be found in [AGP08], 6.4.20. and in [Swi17], 10.1-4.*
- *The spectrum* $\mathbb{S}$ *induce what is called the* ***stable reduced cohomotopy****. It is named that way because the equivalent homology,* $\mathbb{S}_*(-)$ *can be seen as*

$$\mathbb{S}_n(X) \cong \pi_n(\mathbb{S} \wedge X) \cong \pi_n(\Sigma^\infty X) \cong \ colim_k \pi_{n+k}(X \wedge S^k).$$

Notation 2.2.26.
In case of Eilenberg-Mac line spectrum, we will write $HA^*(-)$ *for the induced reduced cohomology and* $H^*(-, A)$ *for the unreduced one.*

Chapter 3

Vector Bundles and MU

Now that we have defined **Sp** and many of its properties. It may be time to define our central notion, the MU spectrum 3.1.27. To do so, we will need to study vector bundles and Thom spaces. If the first is an old notion, dating back to the start of analysis on curves and more generally on manifolds, the latter was introduced by René Thom during the 50s as a generalisation of suspension. Once this is done, we will explain why MU is such an important notion by taking a look at the notion of universal bundles.
Then, we will find an interesting link between geometry on manifolds (especially cobordism) and the MU spectrum using Thom-Pontrjagin cobordism isomorphism.

3.1 Complex Vector Bundles

3.1.1 Central definitions and Thom space

We will give in this section an overview of complex vector bundle theory, and how it relates to some special manifolds, the Grassmanians.

Definition 3.1.1 (Complex Grassmannians)**.**
We define the Grassmannian $G^{\mathbb{C}}_{n,k}$ in 3 equivalent ways. Those different definitions put light on different aspects of the Grassmanian. If the first definition is the most abstract and simple one, the second explains its links to other standard objects while the third puts greater emphasis on its smooth structure.

1.

$$G^{\mathbb{C}}_{n,k} = \left\{K \subset \mathbb{C}^{n+k} |\ K \text{ is a linear subspace of dimension } n \right\}$$

$$A \subset G^{\mathbb{C}}_{n,k} \text{ is open } \iff A = \{K |\ K \subset U, U \text{ open in } \mathbb{C}^{n+k}\}$$

2. *Using $S^n_{\mathbb{C}}$, the n-complex sphere, we define*

$$G_{n,k} = \prod_{i=1}^{n} S_{\mathbb{C}}^{n+k-1}/\sim$$

$$(x_1, \cdots, x_n) \sim (y_1, \cdots, y_n) \iff span\big(x_1, \cdots, x_n\big) = span\big(y_1, \cdots, y_n\big)$$

with standard quotient topology

3. $$G^{\mathbb{C}}_{n,k} = \Big\{M \in Mat_{n+k,n}(\mathbb{C})|\ rank\ M = n\Big\}/\sim$$

$$M \sim N \iff \exists G_1, G_2 \in Gl_n(\mathbb{C}), MG_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_{1,1} & \cdots & \cdots & a_{1,n} \\ \vdots & & & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,n} \end{pmatrix} = NG_2$$

Proposition 3.1.2.

1. $G^{\mathbb{C}}_{1,k} = \mathbb{C}P^k$
2. $G^{\mathbb{C}}_{n,k} \cong G^{\mathbb{C}}_{k,n}$
3. $G^{\mathbb{C}}_{n,k}$ *is a compact smooth manifold without boundary of dimension* $2nk$[1]

Proof of Proposition 3.1.2:

1. From the second definition,
$$G_{1,k} = S^k/\sim = \mathbb{C}P^k$$
2. We use the first definition. Indeed, every vector space K of dimension n has a unique vector space K' of dimension k such that $\mathbb{C}^{n+k} = K \oplus K'$. Thus
$$\varphi : G_{n,k} \cong G_{k,n}$$
$$\varphi(K) = K'$$

[1]We are working with manifolds in $\mathbb{R}$, but it is in fact a nk complex manifold. Also, the fact that it is smooth compact means that it is also a CW complex.

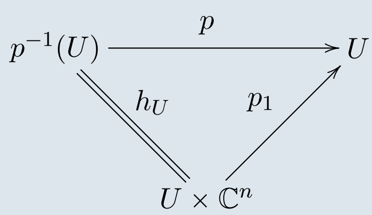
3. From the second definition, we get the compacity part from the fact that S^{n+k-1} is compact. To construct our atlas, we take our third definition and $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant n$. We consider
$$U_{i_1,\cdots,i_n} = \{M \in G^{\mathbb{C}}_{n,k} | \; rk(M_{i_1,\cdots,i_n}) = n\}$$
with $M_{i_1,\cdots,i_n}$ construct using the i lines of M. Then, we define
$$A^{i_1,\cdots,i_n} : U_{i_1,\cdots,i_n} \to \mathbb{C}^{nk}$$
$$A^{i_1,\cdots,i_n}(M) = MM^{-1}_{i_1,\cdots,i_n} \text{ and we send the } k \text{ non trivial lines [2] into } \mathbb{C}^{nk}$$
We have that our maps overlap perfectly on $U_{i_1,\cdots,i_n} \cap U_{j_1,\cdots,j_n}$ and using the rank theorem, we have that every element is in one open of our atlas.
$$\mathcal{A} = \left\{ (U_{i_1,\cdots,i_n}, A^{i_1,\cdots,i_n}) \right\}$$
Thus, we have that $G^{\mathbb{C}}_{n,k}$ is a complex nk compact smooth manifold without boundary and thus a $2nk$ real compact smooth manifold without boundary.

□ 3.1.2

Now, let's define complex vector bundles. In fact, we can define real vector bundles from this definition by simply switching $\mathbb{C}$ for $\mathbb{R}$.

Definition 3.1.3 (Complex vector bundle).
A ***complex vector bundle*** *ξ of dimension n over the topological set B is the couple (E, p) with E a topological set, $p : E \to B$ such that*

1. *$\forall b \in B$, $\exists U$ an open neighbourhood of b and $h_U : p^{-1}(U) \cong U \times \mathbb{C}^n$ an homeomorphism such that the following diagram commute:*



with p_1 being the standard projection for product.

2. *For any two such neighbourhood U, V, $U \cap V \neq \emptyset$, then considering h_1 the restriction of h_U, h_2 the restriction of h_V. Then the composite*
$$(U \cap V) \times \mathbb{C}^n \xrightarrow{h_2{}^{-1}} p^{-1}(U \cap V) \xrightarrow{h_1} (U \cap V) \times \mathbb{C}^n$$
is given by $h_1 \circ h_2^{-1}(b, z) = g_{U,V}(b)(z)$ with $g_{U,V} : U \cap V \to Gl_n(\mathbb{C})$.

[2] That are actually given by the row reduced form

Notation 3.1.4.
For a n-dimensional vector bundle $\xi : E \xrightarrow{p} B$, we call

- *B is the* ***base space*** *of ξ.*
- *E is the* ***total space*** *of ξ.*
- *$\mathbb{C}^n$ is the* ***fibre*** *of ξ.*
- *p is the* ***projection map*** *of ξ.*
- *U_α are called the* ***trivial covering*** *of ξ*
- *$g_{U,V}$ are the* ***transition functions*** *of ξ.*

Example 3.1.5.

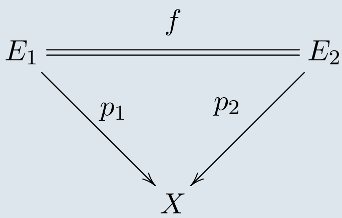
- *Let X be any topological space.* ***the trivial bundle of dimension*** *n* ***over*** *X ($\epsilon^n X$) is defined as $E = X \times \mathbb{C}^n$ and $p : E \to X$ the standard projection with transition functions given by constant maps.*
- *If $n = 1$, then we name a complex vector bundle over X as a* ***line bundle*** *over X.*
- *Let $*$ be a singleton. Then any complex vector bundle over it is trivial, i.e. of the form ϵ^n.*
- *We construct the* ***tautological bundle****. Let the base space be $G^{\mathbb{C}}_{n,k}$ with total space given using our first definition*

$$E = \left\{(x, v) \in G^{\mathbb{C}}_{n,k} \times \mathbb{C}^{n+k} \mid v \in x\right\}$$

$$p(x, v) = x$$

and with trivial transition functions. We denote this bundle of dimension n as $\gamma^{\mathbb{C}}_{n,k}$

Definition 3.1.6 (Vector bundle isomorphism)**.**
Let ξ_1, ξ_2 be complex vector bundles on X. We say that ξ_1 is isomorphic to ξ_2 if $\exists f : E_1 \cong E_2$ and the following diagram commute



Furthermore $f|_{p_1^{-1}(x)} : p_1^{-1}(x) \cong p_2^{-1}(x)$ is such that $f|_{p_1^{-1}(x)} \in Gl_n(\mathbb{C})$. We say it is linear over the fibres.

Now, let's define a few basic tools on vector bundle.

Definition 3.1.7 (Induced complex bundle)**.**
Let ξ be a complex vector bundle over Y of dimension n with total space E and $f : X \to Y$. Then, we define the ***induced bundle*** *$f^*(\xi)$ over X is given by*

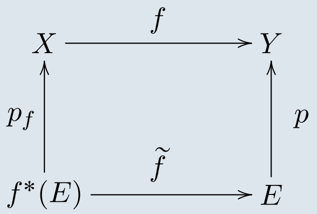
$$f^*(E) = \{(x,e) \in X \times E |\ f(x) = p(e)\}$$

and with projection map $p_f(x,e) = x$. It is also of dimension n.
We also get the continuous projection

$$\widetilde{f} : f^*(E) \hookrightarrow E$$

$$(x,e) \to e$$

and the diagram commute



Verification of Definition 3.1.7:
This is working because for every open U that trivialise E, we have that

$$p^{-1}\big(f^{-1}(U)\big) = \Big\{(x,e) \in X \times E |\ e \in p^{-1}\big(U \cap \mathrm{Im}(f)\big),\ p(e) = f(x)\Big\}$$

But then using $id \times h_U$, we get that

$$p^{-1}\big(f^{-1}(U)\big) \cong \Big\{(x,y,v) \in X \times Y \times \mathbb{C}^n |\ , f(x) = y \in U\Big\} \cong f^{-1}(X) \times \mathbb{C}^n$$

and because we haven't touch to x, the diagram of the definition commute.

□ 3.1.7

Example 3.1.8.
Let ξ be a n-dimensional vector space over X and let $\iota : * \to X$ be the inclusion map. Then

$$\iota^*(\xi) = \epsilon^n$$

Definition 3.1.9 (Sum of complex bundle)**.**
Let ξ_1, ξ_2 be complex vector bundles over X_1 and X_2 with total space E_1, E_2 of dimension n_1, n_2. We define their ***external sum*** *$\xi_1 \times \xi_2$ over $X_1 \times X_2$ given by*

$$E_1 \times E_2 \xrightarrow{p_1 \times p_2} X_1 \times X_2$$

It is a complex vector bundle of dimension $n_1 + n_2$

If $X_1 = X_2 = X$, we define the ***Whitney sum*** *given by*

$$\xi_1 \oplus \xi_2 = \Delta^*(\xi_1 \times \xi_2)$$

with $\Delta : X \to X \times X$ the diagonal map. Its total space is given by

$$E_1 \oplus E_2 = \big\{(x, e_1, e_2) \in X \times E_1 \times E_2 |\ p_1(e_1) = x = p_2(e_2)\big\}$$

Proposition 3.1.10.
Let ξ_1, ξ_2 be bundles on X, $f : Y \to X$, $g : Z \to Y$

1. $$id_X^*(\xi) \cong \xi$$
2. $$(f \circ g)^*(\xi) \cong g^*\big(f^*(\xi)\big)$$
3. $$f^*(\xi_1 \oplus \xi_2) \cong f^*(\xi_1) \oplus f^*(\xi_2)$$

Proof of Proposition 3.1.10:
This is an immediate consequence of the definitions.

□ 3.1.10

Now, let's consider an interesting case of induced bundle on $\gamma_{n,k}$.

Proposition 3.1.11.

- *Let* $i : G^{\mathbb{C}}_{n,k} \to G^{\mathbb{C}}_{n,k+1}$ *be the map induced by* $\iota : \mathbb{C}^{n+k} \to \mathbb{C}^{n+k+1}$. *Then:*

$$i^*(\gamma^{\mathbb{C}}_{n,k+1}) \cong \gamma^{\mathbb{C}}_{n,k}.$$

- *Let* $j : G^{\mathbb{C}}_{n,k} \to G^{\mathbb{C}}_{n+1,k}$ *be the map sending* n *dimensional vector space into* $n+1$ *ones by adding the vector* e_{n+k+1}. *Then:*

$$j^*(\gamma^{\mathbb{C}}_{n+1,k}) \cong \gamma^{\mathbb{C}}_{n,k} \oplus \epsilon$$

with $\epsilon = \epsilon(G^{\mathbb{C}}_{n,k})$

Proof of Proposition 3.1.11:

1. We consider the total space of $i^*(\gamma^{\mathbb{C}}_{n,k+1})$

$$E = \left\{(x, e, v) \in G^{\mathbb{C}}_{n,k} \times G^{\mathbb{C}}_{n,k+1} \times \mathbb{C}^{n+k+1} | v \in e, e = p(e, v) = i(x) = x\right\}$$

$$p(x, e, v) = x$$

But this is[3]

$$E_{n,k} = \left\{(x, v) \in G^{\mathbb{C}}_{n,k} \times \mathbb{C}^{n+k} |\ v \in x\right\}$$

$$p(x, v) = x$$

Therefore,

$$i^*(\gamma^{\mathbb{C}}_{n,k+1}) = \gamma^{\mathbb{C}}_{n,k}$$

2. We consider the total space of $j^*(\gamma^{\mathbb{C}}_{n+1,k})$

$$E = \left\{(x, e, v) \in G^{\mathbb{C}}_{n,k} \times G^{\mathbb{C}}_{n+1,k} \times \mathbb{C}^{n+k+1} | v \in e, e = p(e, v) = j(x) = x \oplus < e_{n+k+1} > \right\}$$

$$p(x, e, v) = x$$

This is equal to

$$E = \left\{(x, v, z) \in G^{\mathbb{C}}_{n,k} \times \mathbb{C}^{n+k} \times \mathbb{C} | v \in e\right\}$$

$$p(x, v, z) = x$$

[3]using the fact that x is contained in $\mathbb{C}^{n+k}$

that can be written as

$$E = \Big\{(x, y_1, y_2, v, z) \in G^{\mathbb{C}}_{n,k} \times G^{\mathbb{C}}_{n,k} \times G^{\mathbb{C}}_{n,k} \times \mathbb{C}^{n+k} \times \mathbb{C} | v \in y_1, y_1 = x = y_2\Big\}$$

$$p(x, y_1, y_2, v, z) = x$$

which is the total space and projection map of $\gamma^{\mathbb{C}}_{n,k} \oplus \epsilon$. Thus,

$$j^*(\gamma^{\mathbb{C}}_{n+1,k}) = \gamma^{\mathbb{C}}_{n,k} \oplus \epsilon.$$

□ 3.1.11

Consider $\mathbb{C}^\infty = \text{colim}_n \mathbb{C}^n$, the sets of all finite sequence in $\mathbb{C}$. We use $\mathbb{C}^\infty$ to define properly the infinite Grassmannian.

Definition 3.1.12 (Infinite Grassmannian and its vector bundle)**.**
Using $i_k : G^{\mathbb{C}}_{n,k} \to G^{\mathbb{C}}_{n,k+1}$, we define the ***infinite k-Grassmannian*** $G^{\mathbb{C}}_n$*:*

$$G^{\mathbb{C}}_n = colim_k\ G^{\mathbb{C}}_{n,k}$$

$$G^{\mathbb{C}}_n = \big\{K \subset \mathbb{C}^\infty |\ K \text{ is a linear subspace of dimension } n\ \big\}$$

Note that if $G^{\mathbb{C}}_n$ is still a CW complex, it is no longer a manifold.
Using $\widetilde{i_k}$ given by 3.1.7, we define the n complex vector bundle

$$\gamma^{\mathbb{C}}_n = colim_k\ \gamma^{\mathbb{C}}_{n,k}.$$

It can be defined as:

$$E = \{(x, v) \in G^{\mathbb{C}}_n \times \mathbb{C}^\infty | v \in x\}$$

$$p(x, v) = x$$

Using $i : G^{\mathbb{C}}_{n,k} \to G^{\mathbb{C}}_n$, we get that

$$i^*(\gamma^{\mathbb{C}}_n) = \gamma^{\mathbb{C}}_{n,k}$$

Similarly to $G^{\mathbb{C}}_{n,k}$, the map $j : G^{\mathbb{C}}_n \hookrightarrow G^{\mathbb{C}}_{n+1}$ given by colimit of $j_k : G^{\mathbb{C}}_{n,k} \hookrightarrow G^{\mathbb{C}}_{n+1,k}$ induce that

$$j^*(\gamma^{\mathbb{C}}_{n+1}) = \gamma^{\mathbb{C}}_n \oplus \epsilon$$

Now, before continuing, we need to define an important notion, paracompactness.

Definition 3.1.13 (Paracompact)**.**
Let X be a topological space. We say it is paracompact if:

1. *X is Hausdorff.*
2. *X is locally compact.*
3. *X is a union of countable union of compact subspaces.*

Example 3.1.14.
The following spaces are paracompact.

- *Compact Hausdorff spaces.*
- *CW complexes.*
- *Metric spaces.*

Proof of those assertion be found in [Hat03], page 35-37.
Now, let's define the notion of partition of unity.

Definition 3.1.15 (Partition of unity)**.**
*Let $X \in$ **Top** and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of M. A* ***partition of unity subordinate to $\mathcal{U}$*** *is a family of continuous maps $\{\psi_\alpha\}_{\alpha \in A}$ with $\psi_\alpha : X \to \mathbb{R}$ such that:*

- $\forall x \in M, \forall \alpha \in A,\ 0 \leqslant \psi_\alpha(x) \leqslant 1$.
- $supp(\psi_\alpha) \subset U_\alpha$.
- $\forall x \in M, \exists\ n_x \in \mathbb{N}$ *such that* $\sum_{\alpha \in A} \psi_\alpha(x) = \sum_{i=1}^{n_x} \psi_{\alpha_i}(x)$.
- $\forall x \in M,\ \sum_{\alpha \in A} \psi_\alpha(x) = 1$.

Paracompacts are very important because they induce partitions of unity.

Theorem 3.1.16.
Let $\{U_\alpha\}$ be an open covering of X a paracompact. Then, $\exists \{\varphi_\alpha\}$ a partition of unity on $\{U_\alpha\}$.

A proof is given in [Bre13], chapter I, 12.11.
Now that we have refreshed ourselves about paracompact, let's go back to our vector bundles.

Definition 3.1.17 (Hermitian metric on complex vector bundle)**.**
An ***Hermitian metric on complex vector bundle*** ξ *with base set* X *is a continuous map*

$$< -,- >: E \times E \to \mathbb{R}$$

such that $\forall x \in X$*,*

$$< -,- >: p^{-1}(x) \times p^{-1}(x) \to \mathbb{R}$$

is a positive inner Hermitian product.

Lemma 3.1.18.
Let X *be a paracompact. Then, for any complex vector bundle* ξ *on* X*, we can give* ξ *an Hermitian metric.*

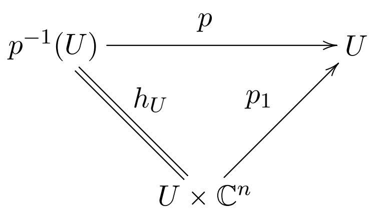
Proof of Lemma 3.1.18:
First, Let's show that every trivial complex vector bundle has a metric. Take

$$p : X \times \mathbb{C}^n \to X$$

Then, for any positive definite Hermitian matrix $\mathbf{H}$ of dimension n, we can define an Hermitian metric given as:

$$< v, w >_T = v^T \mathbf{H} \overline{w}$$

For $v, w \in p^{-1}(x) = \{x\} \times \mathbb{C}^n$
Now, Let ξ be any complex vector bundle on X a paracompact. Then, let's consider U the open subsets of X given by ξ such that the following diagram commute



Then, we can construct on $p^{-1}(U)$ an Hermitian metric given by

$$< v, w >_U = < h_U(v), h_U(w) >_T$$

For $v, w \in p^{-1}(x), x \in U$.

Now, because X is a paracompact, we get that there exists a partition of unity $\{\sigma_U\}$. Then, we can construct our Hermitian metric over E as

$$< e, f >= \sum_U \sigma_U(x) < v, w >_U$$

with $e = (x, v), f = (x, w)$. □ 3.1.18

Now that we have an Hermitian metric on our total space, we can define the following bundles on it.

Definition 3.1.19 (Disk and Sphere bundles)**.**
Let ξ be a complex vector bundle on B equipped with an Hermitian metric. Then, we define the ***disk bundle*** *$D(\xi)$ on B the given by :*

$$E_D = \left\{(x, v) \in E| \ |v| \leqslant 1\right\}$$

$$p = p|_{E_D}$$

We also define the ***sphere bundle*** *$S(\xi)$ on B with:*

$$E_S = \left\{(x, v) \in E| \ |v| = 1\right\}$$

$$p = p|_{E_S}$$

Remark 3.1.20.
For every Hermitian metric we can associate to ξ, they induce the same metric topology because every metric on $\mathbb{C}^n$ are equivalent. Therefore, given two Hermitian metrics on ξ, we have that the $D(\xi)$ and $S(\xi)$ induced would be homeomorphic. Thus, they are only defined by ξ.

Furthermore, if $D(\xi)$ and $S(\xi)$ are fibre bundles, they are not at all vector bundles.

Example 3.1.21.

1. *Consider* $\epsilon^n X$. *Then:*
$$D(\epsilon^n X) = X \times D^n_{\mathbb{C}}$$
$$S(\epsilon^n X) = X \times S^n_{\mathbb{C}}$$

2. *Let* ξ_1, ξ_2 *be two bundles with Hermitian metrics. Then, using the product metric, we get that:*
$$D(\xi_1 \times \xi_2) = D(\xi_1) \times D(\xi_2)$$
$$S(\xi_1 \times \xi_2) = S(\xi_1) \times S(\xi_2)$$

3. *We have that*
$$D(\xi_1 \oplus \xi_2) = D(\xi_1) \times D(\xi_2)/\Delta$$
with
$$\Delta = \{(e, e') \in D(\xi_1) \times D(\xi_2) | p(e) \neq p(e')\}.$$
Similarly,
$$S(\xi_1 \oplus \xi_2) = S(\xi_1) \times S(\xi_2)/\Delta$$
with
$$\Delta = \{(e, e') \in S(\xi_1) \times S(\xi_2) | p(e) \neq p(e')\}.$$

4. *Let* ξ *be a bundle on* B *with Hermitian metric and* $f : X \to B$. $f^*(\xi)$ *inherit the metric form* ξ. *Let* $v, w \in p^{-1}(x)$, *then*
$$< v, w >_f = < \tilde{f}(v), \tilde{f}(w) >$$
with $\tilde{f}(v), \tilde{f}(w) \in p^{-1}(f(x))$. *We can therefore define the disk and sphere bundles. Furthermore, the restriction*
$$\tilde{f} : D(f^*\xi) \hookrightarrow D(\xi)$$
$$\tilde{f} : S(f^*\xi) \hookrightarrow S(\xi)$$
are well defined.

Proposition 3.1.22.
Let ξ *be a complex vector bundle with base space* X *and total space* E. *Then,*

$$X \text{ is Hausdorff compact} \Rightarrow D(\xi), S(\xi) \text{ are compact too.}$$

Proof of Proposition 3.1.22:
Because X is Hausdorff compact, we have an Hermitian metric. Now, let's show that $D(\xi)$ is compact. We will prove this result using the that a space is compact if and only if every infinite net has a convergent subnet.[4]

Let $\{x_\lambda\}_\Lambda$ be an infinite net in E. Then, $\{p(x_\lambda)\}_\Lambda$ is an infinite net in X. Because it is compact, there exists Λ_1 such that $\{p(x_\lambda)\}_{\Lambda_1}$ converge into $x \in X$.
Then, consider $x \in U$ with $p^{-1}(U) \cong U \times D^n_{\mathbb{C}}$. Because $\{p(x_\lambda)\}_{\Lambda_1}$ converge into x, there exist $\overline{\Lambda_1}$ such that $\{p(x_\lambda)\}_{\overline{\Lambda_1}} \subset U$. Thus,

$$\{x_\lambda\}_{\overline{\Lambda_1}} \subset p^{-1}(U) \cong U \times D^n_{\mathbb{C}}$$

Using $\pi : p^{-1}(U) \to D^n_{\mathbb{C}}$ defined using the standard projection, we define $\{\pi(x_\lambda)\}_{\overline{\Lambda_1}}$ a net of $D^n_{\mathbb{C}}$.
By compacity, we get that there exists a subnet λ_2 such that it converge into v.
Thus, we get that $\{x_\lambda\}_{\overline{\Lambda_2}}$ converge into $h_U^{-1}(x, v)$. Thus, $D(\xi)$ is compact.
For $S(\xi)$, we simply see that this a closed set of $D(\xi)$.

□ 3.1.22

Now that we have disk bundles, we want to define something called the Thom space. This is an extension of both the concept of compactification and of the concept of suspension as shown in 3.1.26 and in 3.1.24. It has many very interesting properties.

Definition 3.1.23 (Thom space)**.**
*Let ξ be a complex vector bundle over B equipped with an Hermitian metric. Then, we define the **Thom space** of ξ noted $T(\xi) \in Ob(\mathbf{Top_\bullet})$:*

$$T(\xi) = \big(D(\xi)/S(\xi), S(\xi)\big)$$

*Furthermore, if $f : X \to B$, using the fact that $q : S(f^*\xi) \to S(\xi)$, this induce a map name the **Thomification** of f:*

$$T(f) : T(f^*\xi) \to T(\xi)$$
$$T(f) = \pi \circ \widetilde{f}$$

with π the quotient map.

[4] A proof of this result is in [Pen83]

Proposition 3.1.24.
Let X, Y be topological spaces. ξ_1, ξ_2 are complex vector bundles with Hermitian metric on X, Y

1. $$T(\epsilon^n X) \cong X^+ \wedge S^{2n}_{\mathbb{R}}$$

2. $$T(\xi_1 \times \xi_2) \cong T(\xi_1) \wedge T(\xi_2)$$

3. $$T(\xi_1 \oplus \epsilon^n X) \cong T(\xi_1) \wedge S^{2n}_{\mathbb{R}}$$

Proof of Proposition 3.1.24:

1. First, we have to see that for $n \geqslant 1$, we have

$$D^n_{\mathbb{C}} \cong D^{2n}_{\mathbb{R}}$$

$$S^n_{\mathbb{C}} \cong S^{2n-1}_{\mathbb{R}}$$

$$D^{2n}_{\mathbb{R}}/S^{2n-1}_{\mathbb{R}} \cong S^{2n}_{\mathbb{R}}$$

Using this, we get that

$$\begin{aligned} T(\epsilon^n X) &\cong \Big((X \times D^{2n}_{\mathbb{R}})/(X \times S^{2n-1}_{\mathbb{R}}), (X \times S^{2n-1}_{\mathbb{R}})\Big) \\ &\cong X^+ \times (D^{2n}_{\mathbb{R}}/S^{2n-1}_{\mathbb{R}}) \Big/ X^+ \vee (D^{2n}_{\mathbb{R}}/S^{2n-1}_{\mathbb{R}}, S^{2n-1}_{\mathbb{R}}) \\ &\cong X^+ \wedge S^{2n}_{\mathbb{R}} \end{aligned}$$

2. We write $D(\xi_i)$ as D_i, $S(\xi_i)$ as S_i. Then

$$\begin{aligned} T(\xi_1 \times \xi_2) &= \Big(D_1 \times D_2 \big/ S_1 \times S_2,\ S_1 \times S_2\Big) \\ &\cong \Big((D_1/S_1, S_1) \times (D_2/S_2, S_2) \Big/ (D_1/S_1, S_1) \vee (D_2/S_2, S_2)\Big) \\ &= T(\xi_1) \wedge T(\xi_2) \end{aligned}$$

3. $$D(\xi_1 \oplus \epsilon^n) = D(\xi_1) \times X \times D^n_{\mathbb{C}} \Big/ \Delta$$

$$\text{with } \Delta = \big\{(e, x, v) \in D(\xi_1) \times X \times D^n_{\mathbb{C}} |\ p(e) = x\big\}$$

Thus, we have that

$$D(\xi_1 \oplus \epsilon^n) \cong D(\xi_1) \times D^n_{\mathbb{C}}$$

Similarly, we have that

$$S(\xi_1 \oplus \epsilon^n) = S(\xi_1) \times S^n_{\mathbb{C}}$$

Therefore:

$$\begin{aligned} T(\xi_1 \oplus \epsilon^n) &\cong \left(D(\xi_1) \times D^{2n}_{\mathbb{R}} \big/ S(\xi_1) \times S^{2n-1}_{\mathbb{R}}, S(\xi_1) \times S^{2n-1}_{\mathbb{R}}\right) \\ &\cong \left(\left(D(\xi_1)/S(\xi_1), S(\xi_1)\right) \times (S^{2n}_{\mathbb{R}}, *) \Big/ \left(D(\xi_1)/S(\xi_1), S(\xi_1)\right) \vee (S^{2n}_{\mathbb{R}}, *)\right) \\ &\cong T(\xi_1) \wedge S^{2n}_{\mathbb{R}} \end{aligned}$$

□ 3.1.24

Remark 3.1.25.
Let X be a CW complex, ξ a n complex vector bundle on it. Then, we can give $T(\xi)$ a CW structure.
Given a cell structure on $X, \{e^m_\alpha \mid \alpha \in J_m\}$, then the cell structure on $T(\xi)$ is given by $\{e^{m+2n}_\alpha \mid \alpha \in J_m\}$.
Full details can be found in [Swi17], 12.26.

Lemma 3.1.26.
Let X be a compact Hausdorff space, ξ be a complex vector space on X with total space E. Then, we have that

$$T(\xi) \cong E^\dagger$$

with $E^\dagger = (E \cup \{\infty\}, \infty)$ the one point compactification of E.

Proof of Lemma 3.1.26:
Because every Hausdorff compact space is paracompact, we can define $T(\xi)$. Because X is compact, we have, using proposition 3.1.22, that $D(\xi)$ is compact and therefore so is $T(\xi)$.

Now, we have to see that E is Hausdorff. Indeed, for $e \neq e' \in E$. If $p(e) \neq p(e')$, then $p(e) \in A$, $p(e') \in A'$ and we get $e \in p^{-1}(A)$, $e' \in p^{-1}(A')$, $p^{-1}(A) \cap p^{-1}(A') = \emptyset$. If not, then $e = (x, v'), e' = (x, w)$. Then, because $\mathbb{C}^n$ is Hausdorff, we have that v and w are separated and thus so is e and e'.

Now, consider

$$\alpha : [0,1] \to \mathbb{R}_{\geqslant} \cup \{\infty\}$$

$$\alpha(t) = \begin{cases} tan(\frac{\pi}{2}t) & 0 \leqslant t < 1 \\ \infty & t = 1 \end{cases}$$

We construct

$$f : D(\xi) \twoheadrightarrow E^{\dagger}$$

$$f(e) = \alpha(|e|).e$$

This is a continuous map such that $f : D(\xi)/S(\xi) = T(\xi) \to E^{\dagger}$ is a continuous bijective function from a compact into an Hausdorff space[5]. Thus, f is an homeomorphism.

□ 3.1.26

Now that we know what are Thom space, we can finaly define the MU spectrum.

Definition 3.1.27 (MU spectrum)**.**
Because $G_n^{\mathbb{C}}$ is a CW complex, it has an Hermitian metric. Having $MU(n) = T(\gamma_n^{\mathbb{C}})$, we define the MU ***spectrum:***

$$(MU)_{-n} = \{*\}$$

$$(MU)_{2n} = MU(n)$$

$$MU_{2n+1} = \Sigma MU(n)$$

with

$$p_{2n} = id_{\Sigma MU(n)}$$

$$p_{2n+1} = T(j) : \Sigma^2 MU(n) = T(j^*(\gamma_{n+1})) \hookrightarrow MU(n+1).$$

A good question is why are we so interested in MU. It looks like any other spectra. To give an element of answer, we will have to work on the notion of universal bundle.

3.1.2 Universal bundles

First, we want to show 3.1.30. To do so, we will need the followings propositions that simplify the later work.

Proposition 3.1.28.

1. *A vector bundle $p : E \to X \times [a, b]$ is isomorphic to the trivial bundle if and only if its restriction $p^{-1}(X \times [a, c])$ and $p^{-1}(X \times [c, b])$ are both isomorphic to the trivial for some $c \in [a, b]$.*
2. *For a vector bundle $p : E \to X \times I$, there exists a trivial covering $\{U_\alpha\}$ of X such that*

$$p^{-1}(U_\alpha \times I) \cong U_\alpha \times I \times \mathbb{C}^n.$$

[5]If X is Hausdorff, then so does $X^{\dagger}$

Proof of Proposition 3.1.28:

1. $\Rightarrow$: By definition.

   $\Leftarrow$: We have $E_1 = p^{-1}(X \times [a,c]), E_2 = p^{-1}(X \times [c,b])$, and the homomorphism $h_1 : E_1 \cong X \times [a,c] \times \mathbb{C}^n$, $h_2 : E_2 \cong X \times [c,b] \times \mathbb{C}^n$. Because they may not coincide, we define using

   $$h_1 \circ h_2^{-1} : X \times \{c\} \times \mathbb{C}^n \cong X \times \{c\} \times \mathbb{C}^n$$

   $$q : X \times [c,b] \times \mathbb{C}^n \cong X \times [c,b] \times \mathbb{C}^n$$
   $$q(x,t,v) = (x,t,h_1 \circ h_2^{-1}(v))$$

   Then, we define
   $$h : E \cong X \times [a,b] \times \mathbb{C}^n$$
   $$h = \begin{cases} h_1(x) & x \in E_1 \\ q \circ h_2(x) & x \in E_2 \end{cases}$$

2. $\forall x \in X$, using compactness of I, we can find open neibourhood $U_{x,1}, \cdots, U_{x,k}$ and $0 = t_0 < \cdots < t_k = 1$ such that the bundle is trivial over $U_{x,i} \times [t_i, t_{i+1}]$.

   Then, by using the first point 1) on $U_\alpha = \bigcap_k U_{x,k}$ we get that $U_\alpha \times I$ is trivial.

□ 3.1.28

Lemma 3.1.29.
Let X be a paracompact, then the restriction of $p : E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.

Proof of Lemma 3.1.29:
Let $E_0 = \iota_0^*(E), E_1 = \iota_1^*(E)$. Using proposition 3.1.28, we have an open covering $\{U_\alpha\}$ of X such that E is trivial over $U_\alpha \times I$. Using paracompactness, we can then find a countable subcovering [6] $\{V_i\}$ and a partition of unity $\{\phi_i\}$ on $\{V_i\}$. Thus, we have that E is trivial over $V_i \times I$.

Then, let $\psi_i = \sum_{j=0}^{i} \phi_j$. We define $X_i = \{(x,t) \in X \times I |\ t = \psi(x)\}$ and let $E_i = p^{-1}(X_i)$. We have the following homeomorphism
$$X_i \cong X_{i-1}$$
$$(x, \psi_i(x)) \to (x, \psi_i(x) - \phi_i(x))$$

This induce an isomorphism using the fact that E is trivial on V_i

$$h_i : E_i \cong E_{i-1}$$

[6] This result can be found in [Hat03], lemma 1.21.

$$h_i(x, \psi_i(x), v) = \begin{cases} id & x \in X \backslash V_i \\ (x, \psi_{i-1}(x), v) & x \in V_i \end{cases}$$

Now, using the fact that $\psi_0 = 0$ and $\lim_i \psi_i = 1$,

$$h : E_0 \cong \lim_i E_i$$

$$h = h_0 \circ h_1 \circ \cdots$$

gives us that

$$p^{-1}(X \times \{0\}) \cong p^{-1}(X \times \{1\})$$

□ 3.1.29

Corollary 3.1.30.
Let X be a paracompact, $f, g : X \to Y$ be continuous functions and ξ be a complex vector bundle on Y. Then:

$$f \sim_{Hom} g \Rightarrow f^*(\xi) \cong g^*(\xi)$$

Proof of Corollary 3.1.30:
Let $H : X \times I \to Y$ be the homotopy between f and g. Then, using lemma 3.1.29 on $H^*(\xi)$ we get that

$$f^*(\xi) = \iota_0^*\big(H^*(\xi)\big) \cong \iota_1^*\big(H^*(\xi)\big) = g^*(\xi)$$

□ 3.1.30

Now, we will refocus ourselves on CW complexes. They are by 3.1.14 paracompact, so they follow all previous properties. Lets now define the following functor.

Definition 3.1.31 (Vector bundle contravariant functor)**.**
Let $X, Y \in Ob(\mathbf{HoCW})$, $f \in [X, Y]$. We define the contravariant functor of the n-dimensional complex vector bundle $Vb_n^{\mathbb{C}}$:

$$Vb_n^{\mathbb{C}} : \mathbf{HoCW} \to \mathbf{Set}$$

$$Vb_n^{\mathbb{C}}(X) = Iso\ \{\xi |\ \xi \textit{ is a } n \textit{ complex vector bundles on } X.\ \}$$

$$Vb_n^{\mathbb{C}}([f]) = f^*$$

$$f^*(\xi) = f^*\xi$$

Proposition 3.1.32.
$Vb_n^{\mathbb{C}}$ follows W) and MV) axioms as defined here.

Proof of Proposition 3.1.32:

- W): This comes from the fact that, when considering $\bigvee_\alpha X_\alpha$ and ξ a complex vector bundle on it, $i_\alpha(\xi) \cong p^{-1}(X_\alpha)$ and they muss agree on the base point. Thus, we have that
$$\{i_\alpha^*\} : Vb_n^{\mathbb{C}}(\bigvee_\alpha X_\alpha) \cong \prod_\alpha Vb_n^{\mathbb{C}}(X_\alpha)$$
- MV): Let $A_1 \cup A_2 = X$ and ξ_1 be a bundle on A_1, ξ_2 be a bundle on A_2. Suppose that
$$\Psi : i_{A_2}^*(\xi_1) = p_1^{-1}(A_2) \cong p_2^{-1}(A_1) = i_{A_1}^*(\xi_2)$$
Then, we can construct the wanted complex vector bundle on X as
$$E = E_1 \cup_\Psi E_2$$
$$p = p_1 \cup_\Psi p_2$$

Further details can be found in [Swi17] 11.32.

□ 3.1.32

Corollary 3.1.33.
$\forall n \in \mathbb{N}^*$, there is a unique CW complex (up to homotopy) named $BU(n)$ and a unique complex vector bundle u (up to isomorphism) on it such that
$$T_u : [-, BU(n)] \cong Vb_n^{\mathbb{C}}(-)$$
$$T_u([f]) = [f^*(u)]$$
We name $BU(n)$ the classifying space and u the universal bundle.

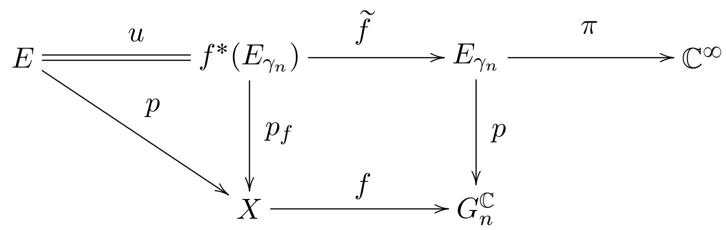
Proof of Corollary 3.1.33:
This is a direct consequence of Brown's representation theorem on sets 2.2.12. The unicity part is aconsequence of 2.2.11.

□ 3.1.33

Now, we want to show that γ_n is the universal bundle.

Theorem 3.1.34.
Consider $G_n^{\mathbb{C}}$. Because it is the colimit of manifold, which are CW complexes, it is itself a CW complex. We also consider γ_n the tautological bundle on $G_n^{\mathbb{C}}$. Then:
$$T_{\gamma_n} : [-, G_n^{\mathbb{C}}] \cong Vb_n^{\mathbb{C}}(-)$$
That is to say, $G_n^{\mathbb{C}}$ is the classifying space and γ_n the universal bundle. Hence, $BU(n) \sim_{Hom} G_n^{\mathbb{C}}$.

Proof of Theorem 3.1.34:
Let ξ be a complex vector of dimension n on X with total space E such that $\xi \cong f^*(\gamma_n)$, with $f : X \to G_n^{\mathbb{C}}$. Then, we have the following commutative diagram:



with $\pi : (x, v) \to v$. Then, we can define

$$g : E \to \mathbb{C}^\infty$$

$$g = \pi \circ \widetilde{f} \circ u.$$

We note that g is a linear injection on each fibers. Inversely, if such a g exists[7], then consider

$$f' : X \to G_n^{\mathbb{C}}$$

$$f'(x) = g(p^{-1}(x))$$

This gives us that

$$E \cong f'^*(E_{\gamma_n})$$

$$\text{using } (x, v) \to (x, g(p^{-1}(x)), v)$$

Furthermore, if g is defined using f, then

$$\begin{aligned} f'(x) &= \pi \circ \widetilde{f} \circ u(p^{-1}(x)) \\ &= \pi \circ \widetilde{f}(p_f^{-1}(x)) \\ &= \pi \circ p^{-1}(f(x)) \\ &= f(x) \end{aligned}$$

Knowing that, let's show that T_{γ_n} is a bijection:

- *Surjective*: Let ξ be a n dimensional vector bundle on $p : E \to X$. Let $\{U_\alpha\}$ be a trivial covering of X. Because it is a paracompact, we can furthermore assume that $\{U_i\}$ is countable[8] and that we have a partition of unity $\{\phi_i\}$. Let

$$g_i : p^{-1}(U_i) \to \mathbb{C}^n$$

$$g_i = p_2 \circ h_{U_i}.$$

[7] i.e. a map that is liner injection on every fibers.
[8] This result can be found in [Hat03], lemma 1.21.

Then, using partition of unity, we extend those maps as

$$\phi_i g_i : E \to \mathbb{C}^n$$

$$\phi_i(x) g_i(x, v) = \phi_i(x) v$$

We use this to define the g map

$$g : E \to (\mathbb{C}^n)^\infty = \mathbb{C}^\infty$$

$$g(x, v) = \big(\phi_0(x) g_0(x, v), \phi_1(x) g_1(x, v), \cdots\big)$$

We see that $g(x, v)$ is a linear injection on every fibres[9]. Therefore,

$$\xi \cong f^*(\gamma_n), \text{ with } f(x) = g(p^{-1}(x)).$$

- *Injective*: Let $f_0, f_1 \in [X, G_n^{\mathbb{C}}]$ such that $f_0^*(\gamma_n) \cong f_1^*(\gamma_n)$. Consider $g_0, g_1 : E \to \mathbb{C}^\infty$. We define the homotopy

$$L_t^o : \mathbb{C}^\infty \times I \to \mathbb{C}^\infty$$

$$L_t^o(x_1, x_2, x_3, \cdots) = t(x_1, x_2, x_3, \cdots) + (1 - t)(x_1, 0, x_2, 0, x_3, \cdots)$$

and consider $g_0' = L_1^o \circ g_0$, $g_1' = L_1^e \circ g_1$. We have that

$$g_0' \sim_{Hom} g_0, g_1' \sim_{Hom} g_1.$$

We define

$$H_t = t g_0' + (1 - t) g_1'.$$

This gives us an homotopy g_t between g_1 and g_2 such that g_t is linear and injective on the fibres.

Then, consider $f_t = g_t(p^{-1}(x))$. It is such that

$$f_0 = f_0' = \iota_0 \circ f_t, \; f_1 = f_1' = \iota_1 \circ f_t$$

□ 3.1.34

Corollary 3.1.35.
Let $X \in Ob(\mathbf{CW})$ and ξ be a n-dimensional complex vector bundle on it. Then

$$\exists f : T(\xi) \to MU(n)$$

Proof of Corollary 3.1.35:
Thanks to theorem 3.1.34, $\exists j : X \to G_n^{\mathbb{C}}$ such that $\xi \cong j^*(\gamma_n)$. Then, it simply suffice to take $f = T(j)$.

□ 3.1.35

[9] That is with x fixed

3.2 Cobordism

Now that we know why MU is not a standard spectrum, we want to find, thanks to 2.2.24, which cohomology and homology it defines. Answering this question is the goal of the following section.
To do so, we will give a description of a structure on manifolds named cobordism and show its link with MU.

3.2.1 Reminders on manifolds

Before starting definition of cobordism, we remind ourselves of some constructions and properties of smooth manifolds. On any manifold M, there exists the tangent bundle TM, a real vector bundle defined as

$$TM = \Big\{(x, v) \in M \times \mathbb{R}^n | \ v \in T_x M\Big\}$$

Full details on tangent space and bundle can be found in [Kos13], 1.4 and 1.5.
But we can give a manifold another kind of bundle, namely the normal bundle

Definition 3.2.1 (Normal bundles)**.**
Let W be a smooth manifold of dimension m and M be an n dimensional embedded submanifold of W. Then, we define the ***normal bundle****:*

$$\mathbf{N}(M, W) = \Big\{(x, v) \in M \times TW | \ v \in T_x W \ \text{ and } v \perp \ T_x M\Big\}$$

$$p(x, v) = x$$

The trivial covering is given by the atlas on our manifold W.
We have that $\mathbf{N}(M, W)$ is of dimension $m - n$.

In case that M is a manifold with boundary, then, we define $\mathbf{N}(M, W) = \mathbf{N}(int(M), W)$ extended by continuity onto its boundary.

Proposition 3.2.2.

1. *The normal bundle is a smooth manifold of dimension m.*
2. *If $W = \mathbb{R}^m$, then, using $\iota : \mathbb{R}^m \hookrightarrow \mathbb{R}^{m+1}$, we get*

$$\mathbf{N}(M, \mathbb{R}^{m+1}) \cong \mathbf{N}(M, \mathbb{R}^m) \oplus \epsilon_{\mathbb{R}}(M)$$

This interesting construction becomes essential in manifold theory using the following 2 theorems.

Theorem 3.2.3 (Whitney embedding Theorem).
Let M be a n dimensional smooth manifold. Then M can be embedded into $\mathbb{R}^{2n+1}$.

Proof in [Bre13], chapter II, 10.08.

Theorem 3.2.4 (Tubular neighbourhood theorem).
Let W be a m dimensional smooth manifold, M a n dimensional embedded compact submanifold. Then, $\exists T$ open neighbourhood of M such that

$$T \cong \mathbf{N}(M, W)$$

Furthermore, M is the zero section of this diffeomorphism.

Proof in [Bre13], chapter II, 11.04 and 11.14.

There also exists a variant of the tubular neighbourhood for boundary of manifolds, named in this case **collars**.

Theorem 3.2.5 (Differential collaring theorem).
Let W be a compact smooth m manifold with boundary. Then, $\mathbf{N}(\partial W, W) = \epsilon_{\mathbb{R}}(\partial W)$ and $\exists T$ an open neighbourhood of ∂W such that

$$T \cong \partial W \times \mathbb{R}_{\geqslant 0}$$

with ∂W the zero section of this diffeomorphism.

Using those theorems, we can define some important operations on manifolds that are powerful tools when handling manifolds.

Definition 3.2.6 (Smooth product)**.**
*Let M, N be smooth manifold with boundary with respective dimension m, n. Then, we define the **smooth product** as giving a smooth structure on the $M \times N$ such that:*

- $p : M \hookrightarrow M \times N$, $p : N \hookrightarrow M \times N$ *are embedding.*
- $\partial(M \times N) = \partial M \times N \cup M \times \partial N$.

Definition 3.2.7 (Gluing)**.**
Let V, W be smooth compact $n+1$ manifolds with boundary and M be a n smooth compact manifold with boundary such that it is embedded into ∂V and ∂W. Then, we define the gluing of V and W alongside M as a smooth structure on $V \cup_M W$, written

$$V \#_M W.$$

- $V \#_M W$ *is a smooth compact $n+1$ manifold with boundary.*
- $V \backslash M$, $W \backslash M$ *are smooth submanifolds.*
- $\iota : M \hookrightarrow V \#_M W$ *is an embedding.*
- $\partial(V \#_M W) = \partial V \backslash M \bigsqcup \partial W \backslash M$.

We also need a theorem of approximation on smooth functions.

Theorem 3.2.8 (Smooth homotopy theorem)**.**

1. *Let $f : M \to N$ be a continuous function between 2 manifolds. Then,*

$$\exists \widetilde{f} : M \to N \text{ a smooth map such that } f' \sim_{Hom} f$$

2. *Let $H : M \times [0,1] \to N$ be an homotopy between 2 smooth maps f and g. Then,*

$$\exists \widetilde{H} : M \times [0,1] \to N \text{ a smooth homotopy between } f \text{ and } g$$

[Bre13], chapter II, 11.8. and [tD08] 15.8.4.

3.2.2 Complex cobordism

Now that we have reminded ourselves of important results on manifolds, we can define what is called complex cobordism. The main issue we have is that manifold with boundary are not well defined for complex manifold. To solve this problem, we define the following structure on smooth manifolds.

Definition 3.2.9 (Stably complex manifolds)**.**
Let M be a smooth k manifold. We say that M is ***stably complex*** *if for some $n \in \mathbb{N}$, there exists an isomorphism such that*

$$\mathbf{N}(M, \mathbb{R}^{2n+k}) \cong \xi$$

with ξ a n complex vector bundle. We usually note this (M, ξ).
In case M has a boundary, we say it is stably complex if $int(M)$ is.

Remark 3.2.10.

1. Every complex manifolds of dimension n, seen as $2n$ real manifold, are stably complex.
2. If $\mathbf{N}(M, \mathbb{R}^{2n+k})$ is complex, then $\mathbf{N}(M, \mathbb{R}^{2(n+1)+k}) = \mathbf{N}(M, \mathbb{R}^{2n+k}) \oplus \epsilon_{\mathbb{R}}^2 \cong \xi \oplus \epsilon_{\mathbb{C}}$
3. Because for any manifold M, $TM \oplus \mathbf{N}(M, \mathbb{R}^{2n+k}) \cong \epsilon_{\mathbb{R}}^{2n+k}(M)$, we get that we can give for some u $TM \oplus \epsilon_{\mathbb{R}}^u(M)$ a complex structure.

We can now define complex bordism.

Definition 3.2.11 (Unitary cobordism)**.**
Let $X \in Ob(\mathbf{CW})$. Let $(M, \xi_M), (N, \xi_N)$ be n compact stably complex smooth manifolds, $f : M^+ \to X$, $g : N^+ \to X$. We say that (M, ξ_M, f) is ***unitary cobordant*** *to (N, ξ_N, g) if there exists (W, ξ_W, F) with W a $n+1$ compact stably complex manifold with boundary embedded in $\mathbb{R}^{2w+n+1}$ and $F : W^+ \to X$ such that:*

1. *$\partial W = M \sqcup N$*
2. *$F|_M = f, F|_N = g$*
3. *$\mathbf{N}(M, \mathbb{R}^{2w+n+1}) \cong \iota_M^*(\xi_W) \oplus \pm\epsilon_{\mathbb{R}} \cong \xi_M \oplus \epsilon_{\mathbb{C}}^u \oplus \pm\epsilon_{\mathbb{R}}$ with $\epsilon_{\mathbb{R}}$ given by the induced orientation on ∂W.*
4. *$\mathbf{N}(N, \mathbb{R}^{2w+n+1}) \cong \iota_N^*(\xi_W) \oplus \mp\epsilon_{\mathbb{R}} \cong \xi_N \oplus \epsilon_{\mathbb{C}}^v \oplus \mp\epsilon_{\mathbb{R}}$ with $\epsilon_{\mathbb{R}}$ also given by the induced orientation on ∂W.*

In 3. and 4., the isomorphisms are such that $\forall x \in M$ or N, $\varphi|_{p^{-1}(x)} \in Gl^{+}_{2w+1}(\mathbb{R})$. The $\epsilon^u_{\mathbb{C}}$ are used to make sure that dimension agrees.

Unitary cobordism is an equivalence relation, written $\sim_{Cob}$.

Verification of Definition 3.2.11:

1. *identity:* For (M, ξ, f) with $N(M, \mathbb{R}^{2n+k}) \cong \xi$, consider $M \times [0,1]$ with $F(x,t) = f(x)$. We have to show that $M \times [0,1]$ is stably complex and that vector bundles agree. This comes from that $M \times (0,1) \cong M \times \mathbb{R}$. Then, we see that
$$\mathbf{N}(M \times \mathbb{R}, \mathbb{R}^{2n+k+1}) \cong \mathbf{N}(M, \mathbb{R}^{2n+k}) \times \mathbb{R} \cong \xi \times \mathbb{R}$$
This is an abuse of notation but we see it is is a complex vector bundle.
We have that
$$\mathbf{N}\big(M \times \{0\}, \mathbb{R}^{2n+k+1}\big) \cong \mathbf{N}(M, \mathbb{R}^{2n+k}) \oplus \epsilon_{\mathbb{R}} \cong \xi \oplus \epsilon_{\mathbb{R}}$$
and it is well oriented. Similarly,
$$\mathbf{N}\big(M \times \{1\}, \mathbb{R}^{2n+k+1}\big) \cong \mathbf{N}(M, \mathbb{R}^{2n+k}) \oplus -\epsilon_{\mathbb{R}} \cong \xi \oplus -\epsilon_{\mathbb{R}}.$$
Thus, using $(M \times [0,1], F)$, we get $(M, \xi, f) \sim_{Cob} (M, \xi, f)$

2. *reflexivity:* By definition.

3. *Associativity:* Let (W_1, ξ_1, F_1) be the cobordism between (M, ξ_M, f) and (N, ξ_N, g), (W_2, ξ_2, F_2) be the cobordism between (P, ξ_P, h) and (N, ξ_N, g). Then, consider
$$W_1 \#_N W_2$$
$$F = F_1 \cup_N F_2.$$
This gives us the wanted bordism, but we still need to show that $W_1 \#_N W_2$ is stably complex. For $int(W_1 \#_N W_2) = \dot{W}_1 \sqcup \dot{W}_2 \sqcup N$. Using 3.2.5, there exists $Y = Y_1 \cup_N Y_2$ an open of $W_1 \#_N W_2$ such that $Y \cong \mathbf{N}(N, W_1) \cong \mathbf{N}(N, W_2)$. Then, using the fact that
$$\mathbf{N}\big(\mathbf{N}(N, W), \mathbb{R}^u\big) \cong \iota_N^* \mathbf{N}(W, \mathbb{R}^u)$$
We see that $\exists n \in \mathbb{N}$
$$N(int(W_1 \#_N W_2), \mathbb{R}^{2n+k+1}) = N(\dot{W}_1, \mathbb{R}^{2n+k+1}) \cup N(\dot{W}_2, \mathbb{R}^{2n+k+1}) \cup N(Y, \mathbb{R}^{2n+k+1})$$
which has a complex structure because union of complex vector bundle that are the same on their intersection. The fact that boundary conditions are respected comes from the fact that we have not changed in a substantive way the boundary (at worst, we have added an even number of trivial bundles). Thus, we get that
$$(M, \xi_M, f) \sim_{Cob} (P, \xi_P, f)$$

□ 3.2.11

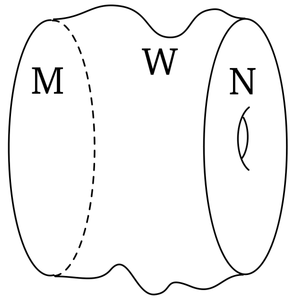


Figure 3.1: Cobordism W between $M = S^2_{\mathbb{R}}$ and $N = (S^1_{\mathbb{R}} \times S^1_{\mathbb{R}})$ [Bar09].

Having this equivalence relation, we use it to construct the complex bordism group.

Definition 3.2.12 (Unitary bordism group)**.**
Let $X \in Ob(\mathbf{CW})$. We define the n ***unitary bordism group*** *on X as*

$$\Omega_n^U(X) = \Big\{(M, \xi_M, f) | M \textit{ compact stably complex } n\textit{-manifold }, f : M^+ \to X\Big\}/_{\sim_{Cob}}$$

$$[M, \xi_M, f] + [N, \xi_N, g] = [M \sqcup N, \xi_M \sqcup \xi_N, f \sqcup g].$$

Thanks to Whitney theorem 3.2.3, this is a set[10]*. In fact, $\Omega_n^U(X)$ is an abelian group.*

From now on, we will drop writing ξ_M in (M, ξ_M, f) when it is clear from context.

Verification of Definition 3.2.12:
To show that the addition is well defined, consider (W_M, F) a cobordism between (M_1, f_1) and (M_2, f_2), (W_N, G) a cobordism between (N_1, g_1) and (N_2, g_2). Then,

$$(W_M \sqcup W_N, F \sqcup G) \text{ is the cobordism between } (M_1 \sqcup N_1, f_1 \sqcup g_1) \text{ and } (M_2 \sqcup N_2, f_2 \sqcup g_2)$$

Every other axioms of group is quite easy to show. If the associativity and commutativity are trivial, We can find the zero by considering $0 = [\emptyset, \emptyset, f : * \to X]$. The inverse is a bit more tricky. For $[M, \xi_M, f]$ with ξ n dimensional, we define $[M, \overline{\xi_M}, f]$ with $\overline{\xi_M}$ given by

$$\iota_U^*(\overline{\xi_M}) \cong U \times \mathbb{C}^{n-1} \times \overline{\mathbb{C}}^{11}$$

Then, the cobordism $(M \times [0,1], \xi_\times, F)$ as defined in proof of 3.2.11 is such that us that

$$\mathbf{N}(M \times \{1\}, \mathbb{R}^{2n+k+1}) \cong \xi_M \oplus \pm\epsilon_{\mathbb{R}} \cong \overline{\xi_M} \oplus \mp\epsilon_{\mathbb{R}}$$

[10]Indeed, we have that the set of all n-manifold can be seen as a subset of $\mathcal{P}(\mathbb{R}^{2n+1})$.
[11]Using complex conjugate $\overline{x + iy} = x - iy$.

Thus, $(M \times [0,1], \xi_\times, F)$ can be seen as a cobordism between $(M \sqcup M, \xi_M \sqcup \overline{\xi_M}, f \sqcup f)$ and $[\emptyset, \emptyset, f : * \to X] = 0$. So $[M, \overline{\xi_M}, f]$ is the inverse of $[M, \xi_M, f]$.

□ 3.2.12

Definition 3.2.13.
We can in fact see $\Omega_n^U(X)$ as a functor:

$$\Omega_n^U : \mathbf{CW} \to \mathbf{Ab}$$

$$\textit{with } \Omega_n^U(f) : \Omega_n^U(X) \to \Omega_n^U(Y)$$
$$\Omega_n^U(f)\big([M, \xi_M, g]\big) = [M, \xi_M, f \circ g]$$

Proposition 3.2.14.
If $f \sim_{Hom} g$, Then $\Omega_n^U(f) = \Omega_n^U(g)$

Proof of Proposition 3.2.14:
Let $h : X \times [0,1] \to Y$ be the homotopy between f and g. Then, $\forall [M, \xi_M, \phi] \in \Omega_n^U(X)$, we get that

$$[M, \xi_M, f \circ \phi] = [M, \xi_M, g \circ \phi]$$

For that, we use the cobordism

$$\Big(M \times [0,1], \xi_\times, h(\phi(x), t)\Big)$$

and using $\xi_\times$, same complex vector bundle as before. □ 3.2.14

Theorem 3.2.15.
Let

$$\Omega_n^U : \mathbf{HoCW} \to \mathbf{Ab}$$

Then, Ω_n^U is an unreduced homology theory that follows the wedge and the WHE axiom as defined in 1.3.2.

The following paper [Hop16] is a proof this theorem.
Now that we have an homotopy theory, we want to find the spectra it is equivalent to.

3.2.3 Thom-Pontrjagin cobordism isomorphism

Here is a small result that will help us simplify the structure of bordism.

Proposition 3.2.16.
Let (W, F) be a cobordism between (M, f) and (N, g), both beeing embedded in $\mathbb{R}^{2n+k}$. Then, we can consider W as embedded into $\mathbb{R}^{2n+k} \times [0,1]$ with $W \cap \mathbb{R}^{2n+k} \times \{0\} = M$, $W \cap \mathbb{R}^{2n+k} \times \{1\} = N$.

Proof of Proposition 3.2.16:
Because manifolds are metric spaces, we can use Urysohn lemma on the closed sets M and N. Thus, we get

$$\alpha : W \to [0,1]$$

$$\alpha(M) = 0, \alpha(N) = 1 \text{ and } \alpha \text{ continuous}$$

Then, using smooth approximation theorem, we can assume that α is a smooth map. Now, because 0 and 1 are regular points, we get an open neighbourhood around 0 and 1. Now, consider t_1 the smallest singular value and t_2 the biggest singular value. Consider

$$W' = \alpha^{-1}\big([0, t_1) \sqcup (t_2, 1]\big)$$

This is a $k+1$ submanifold with boundary ∂W. Then, because W is stably complex, then

$$\mathbf{N}(W', \mathbb{R}^{2n+k+1}) \cong \iota^* \mathbf{N}(W, \mathbb{R}^{2n+k+1}) \cong \iota^* \xi$$

and is thus stably complex and preserve boundary conditions. Then, we have that $(W', F|_{W'})$ is still a bordism. Furthermore, we define an embedding

$$f : W' \hookrightarrow \mathbb{R}^{2n+k} \times [0,1]$$

$$f(x) = (\iota_\alpha(x), \alpha(x))$$

with ι_α the embedding on $\alpha^{-1}(t)$.

□ 3.2.16

Remark 3.2.17.
We consider

$$\Omega^U_{n,k}(X) = \{(M, f) |\ M \subset \mathbb{R}^{2n+k}, f : M^+ \to X\} /_{\sim_{Cob}}$$

with $(M, f) \sim_{Cob} (N, g)$ if $\exists W$ a cobordism with $W \subset \mathbb{R}^{2n+k} \times [0,1]$.
We can see that $\Omega^U_{n,k}(X)$ is an abelian group and furthermore, we have that

$$\Omega^U_k(X) = \operatorname{colim}_n \Omega^U_{n,k}(X)$$

with injection given by embedding $i : \mathbb{R}^{2n+k} \hookrightarrow \mathbb{R}^{2(n+1)+k}$.

Now, we define the cornerstone of this subsection.

Definition 3.2.18 (Thom-Pontrjagin construction)**.**
Let M be a compact stable complex manifold. Consider T given by tubular neighbourhood 3.2.4. We have $\varphi : T \cong N(M, \mathbb{R}^{2n+k}) \cong \xi$. Then, seeing ξ as $int\big(D(\xi)\big)$, we extend this function onto $\mathbb{R}^{2n+k}$ by sending everything not in T on the sphere bundle. Using Thom space, we get

$$\overline{\varphi} : S^{2n+k} \to T(\xi)$$

But then, using 3.1.35, we get what is called the ***Thom-Pontrjagin construction****:*

$$\Phi_M : S^{2n+k} \xrightarrow{\varphi} T(\xi) \xrightarrow{T(j)} MU(n)$$

Thom-Pontrjagin construction has some very good properties.

Proposition 3.2.19.

1. *This map is unique up to homotopy for a given M.*
2. *If we consider $\xi \cong \mathbf{N}(M, \mathbb{R}^{2n+k})$, then the Thom-Pontrjagin construction given on $\mathbf{N}(M, \mathbb{R}^{2(n+1)+k})$ is simply $\Sigma^2 \Phi_M$.*
3. *If $(M, f) \sim_{Cob} (N, g)$, then $\Phi_M \sim_{Hom} \Phi_N$.*

Proof of Proposition 3.2.19:

1. First, on the choice of T, we have that they are all diffeomorphic to each others. Consider T_1, T_2 two tubular neighborhood and let $H : \mathbf{N}(M, \mathbb{R}^{2n+k}) \times [0,1] \to M$ be the homotopy equivalence by retraction on the zero section. Then, consider the following map.

$$\Delta(x,t) = \begin{cases} \varphi_1 \circ H(\varphi_1^{-1}(x), 2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \varphi_2 \circ H(\varphi_1^{-1}(x), 2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

and $E_t = \Delta(Y_1, t)$. Then, consider

$$p_t : E_t \hookrightarrow \mathbf{N}(N, \mathbb{R}^{2n+k})$$

$$p_t = \begin{cases} \varphi_1 & 0 \leqslant t \leqslant \frac{1}{2} \\ \varphi_2 & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Then, by extending p_t to all $\mathbb{R}^{2n+k}$, we get the homotopy between $\overline{\varphi_1}$ and $\overline{\varphi_2}$

Now, for the second part, it is a consequence of 3.1.34. j is unique on homotopy and thus so does $T(j)$.

2. We have that $\mathbf{N}(M, \mathbb{R}^{2(n+1)+k}) \cong \mathbf{N}(M, \mathbb{R}^{2n+k}) \oplus \epsilon^2_{\mathbb{R}}(X)$. Thus, if T is the tubular neighbourhood of $\mathbf{N}(M, \mathbb{R}^{2n+k})$, then $T' = T \times \mathbb{R}^2$ is a tubular neighbourhood of $\mathbf{N}(M, \mathbb{R}^{2(n+1)+k})$. Thus $\overline{\varphi}' = \overline{\varphi} \wedge id_{S^2}$. Therefore, the Thom-Pontrjagin construction gives us

$$S^{2(n+1)+k} \xrightarrow{\overline{\varphi} \wedge id_{S^2}} T(\xi \oplus \epsilon_{\mathbb{C}}) = T(\xi) \wedge S^2_{\mathbb{R}} \xrightarrow{T(j) \wedge id_{S^2}} MU(n) \wedge S^2_{\mathbb{R}}$$

   And this just $\Sigma^2 \Phi_M$

3. Let $W \subset \mathbb{R}^{2n+k} \times [0,1]$ be a cobordism between M and N given by 3.2.16. Then, we apply the Thom-Pontrjagin construction on W[12], extended on M and N gives us a map

$$\Phi_W : S^{2n+k} \times [0,1] \to MU(n)$$

   With $\Phi_W \circ i_0 = \Phi_M$ and $\Phi_W \circ i_1 = \Phi_N$. That is to say, we have

$$\Phi_M \sim_{Hom} \Phi_N.$$

□ 3.2.19

The Thom-Pontrjagin gives us therefore a morphism.

Definition 3.2.20 (Thom-Pontrjagin morphism)**.**
Let $X \in Ob(\mathbf{HoCW})$, Then, we define the ***Thom-Pontrjagin morphism:***

$$\Phi : \Omega^U_{n,k}(X) \to \pi_{2n+k}(X^+ \wedge MU(n))$$

$$[M, f] \to [\Phi_M]$$

Verification of Definition 3.2.20:
This is indeed a group morphism because

$$\mu(f,g)^*(\gamma_n) \cong f^*(\gamma_n) \sqcup g^*(\gamma_n)$$

With μ as defined on 1.2.6. Then,

$$\Phi_{M \sqcup N} : S^{2n+k} \xrightarrow{\mu(\varphi_1, \varphi_2)} T(\xi_1) \sqcup T(\xi_2) \xrightarrow{T(j_1) \sqcup T(j_2)} MU(n)$$

which is just $\mu(\Phi_M, \Phi_N)$.
Furthermore, $\Phi_{\emptyset} : S^{2n+k} \to *$ is the 0 of $\pi_{2n+k}(X \wedge MU(n))$.

□ 3.2.20

In fact, this morphism can be naturally extended into a natural homology transformation

[12] To be exact, we define it on W_t with $t \in (0,1)$ fixed and take the union of the tubular neighbourhoods.

Definition 3.2.21 (Thom-Pontrjagin transformation)**.**
*Let $X \in Ob(\mathbf{HoCW})$. Then, we define the **Thom-Pontrjagin transformation**:*

$$\Phi : \Omega_k^U(X) \to \pi_k(X^+ \wedge MU)$$

$$[M, f] \to [\Sigma^{-2n}\Sigma^{\infty}S^{2n+k}, \Sigma^{-2n}\Sigma^{\infty}\Phi_M]$$

seeing $\Sigma^{-2n}\Sigma^{\infty}S^{2n+k}$ as a cofinal subspectrum of $\Sigma^k\mathbb{S}$.

To go further on this natural transformation, we need to do a small detour on the notion of transversality.

Definition 3.2.22 (Transversal maps)**.**
*Let $f : M \to N$, $g : V \to N$ be smooth maps. We say that f **is transversal to** g if whenever $f(p) = g(q)$*

$$Df(T_pM) + Dg(T_qV) = T_{f(p)}N$$

with Df the smooth pushforward. We note transversality as $f \pitchfork g$.

Proposition 3.2.23.
Let $f : M \to N$, $g : V \to N$, $f \pitchfork g$. Then $f^{-1}\big(g(V)\big)$ is a regular submanifold of M.

proof given by [Kos13] in chapter IV, 1.4.

Theorem 3.2.24 (Thom approximation theorem)**.**
Let $f : M \to N, g : V \to N$ be two smooth maps.

$$\exists \widetilde{f} : M \to N, \widetilde{f} \sim_{Hom} f,\ \widetilde{f} \pitchfork g$$

proof given by [Kos13] in chapter IV, 2.5.

Now, equip with this theorem, we can prove the Thom-Pontrjagin isomorphism.

Theorem 3.2.25 (Thom-Pontrjagin isomorphism)**.**

$$\Phi : \Omega^U_{k,n}(*) \cong \pi_{2n+k}(MU(n) \wedge S^0)$$

Proof of Theorem 3.2.25:
First, $\pi_{2n+k}(MU(n) \wedge S^0) = \pi_{2n+k}(MU(n))$.
In this proof, we will use two small observations. First, we have that $\gamma_n = \operatorname{colim}_k \gamma_{n,k}$. This insure that $MU(n)$ can be considered as

$$MU(n) = \operatorname{colim}_k T(\gamma_{n,k})$$

with injection given by $T(i_k) : T(\gamma_{n,k}) \hookrightarrow T(\gamma_{n,k+1})$.

Furthermore, let's consider $\gamma_{n,k}$. We can give its total space $\left\{(x, v) \in G^{\mathbb{C}}_{n,k} \times \mathbb{C}^{n+k} \mid v \in x\right\}$ the structure of a $k(n+1)$ complex manifold[13]. Indeed, it is locally equivalent to $U \times \mathbb{C}^n$ with U open subset of $G^{\mathbb{C}}_{n,k}$ and thus also a manifold.

Now, let's show Φ is an isomorphism.

- *surjective*: Let $f : S^{2n+k} \to MU(n)$. Then, using 2.2.21, we get that f is defined by a map
$$f : S^{2n+k} \to T(\gamma_{n,j}).$$
Because $G^{\mathbb{C}}_{n,j}$ is compact, 3.1.26 gives us that $T(\gamma_{n,j})$ is just the one point compactification of its total space E. We therefore consider the map
$$f|_{f^{-1}(E)} : U \to E_{\gamma_{n,j}}$$
with $U \subset \mathbb{R}^{2n+k}$. It is thus a stably complex manifold. Then, using 3.2.8, we get $\overline{f} : U \to E_{\gamma_{n,j}}$ a smooth map such that
$$\overline{f} \sim_{Hom} f$$
Using 3.2.24, we finally get
$$\widetilde{f} : U \to E_{\gamma_{n,j}}$$
$$\widetilde{f} \pitchfork G^{\mathbb{C}}_{n,j}$$
$$\widetilde{f} \sim_{Hom} \overline{f}$$
Now, consider $M = f^{-1}(G^{\mathbb{C}}_{n,j})$. By 3.2.23, M is an embedded manifold in $\mathbb{R}^{2n+k}$.

[13] So it is a $2k(n+1)$ stably complex manifold

By thinking a bit at $\mathbf{N}(G^{\mathbb{C}}_{n,j}, E_{\gamma_{n,j}})$, we see that it is just $\gamma_{n,j}$ by the structure of the manifold $E_{\gamma_{n,j}}$. Furthermore, by 3.2.4, $G^{\mathbb{C}}_{n,j}$ has a tubular neighbourhood and it is just $E_{\gamma_{n,j}}$.

Using this, we get that:

1. $$\mathbf{N}(M, \mathbb{R}^{2n+k}) = \mathbf{N}(M, U) \cong f^*\mathbf{N}(G^{\mathbb{C}}_{n,j}, E_{\gamma_{n,j}}) \cong f^*\gamma_{n,j}.$$

   Thus, M is stably complex.
2. $U = f^{-1}(E_{\gamma_{n,j}})$ is a tubular neighbourhood for $M = f^{-1}(G^{\mathbb{C}}_{n,j})$.

Now, if we consider Φ_M, we get that

$$\begin{array}{c} \overset{f}{\overbrace{\hphantom{S^{2n+k} \xrightarrow{\varphi} T\big(\mathbf{N}(M, \mathbb{R}^{2n+k})\big) \xrightarrow{T(i\circ \widetilde{f})} MU(n)}}} \\ S^{2n+k} \xrightarrow{\varphi} T\big(\mathbf{N}(M, \mathbb{R}^{2n+k})\big) \xrightarrow{T(i\circ \widetilde{f})} MU(n) \end{array}$$

Thus, $\Phi_M \sim_{Hom} f$.

- *injectivity*: Let $H : S^{2n+k} \times [0,1] \to MU(n)$ be an homotopy between Φ_M and Φ_N. Then, using a similar reasoning than for surjectivity, we get a smooth map

  $$\widetilde{H} : U \times [0,1] \to E_{\gamma_{n,j}}$$

  $$\widetilde{H} \pitchfork G^{\mathbb{C}}_{n,j}$$

  Then, $W = \widetilde{H}^{-1}(G^{\mathbb{C}}_{n,j})$ gives us a bordism between M and N. Furthermore,

  $$\mathbf{N}(\dot{W}, \mathbb{R}^{2n+k+1}) \cong \widetilde{H}^*\mathbf{N}(G^{\mathbb{C}}_{n,j}, E_{\gamma_{n,j}}) = \widetilde{H}^*\gamma_{n,j}$$

  Thus, W is stably complex.

□ 3.2.25

Corollary 3.2.26 (Thom-Pontrjagin equivalence)**.**

$$\Phi : \Omega^U_k(X) \cong \pi_k(X^+ \wedge MU)$$

i.e., we have found a representation of the unreduced homology induced by MU.

Proof of Corollary 3.2.26:
We are working with unreduced cohomology. Due to the duality between reduced and unreduced showed in 1.3.7, we see that 1.4.24 can be applied in the unreduced case. It thus suffice to show

$$\Phi : \Omega_k^U(*) \cong \pi_k(S^0 \wedge MU)$$

To get this result, we see that $\pi_k(S^0 \wedge MU) = \pi_k(MU)$ and using previous theorem 3.2.25, we get

$$\begin{array}{ccc} \Omega_k^U(*) & \xrightarrow{\quad\Phi\quad} & \pi_k(MU) \\ \| & & \| \, \alpha \\ \operatorname{colim}_n \, \Omega_{k,n}^U(*) & \overset{\Phi}{=\!=\!=} & \operatorname{colim}_n \, \pi_{2n+k}(MU(n)) \end{array}$$

□ 3.2.26

It is interesting to note that the computation of $MU_*(-)$ we have found is quite geometric, because it study manifolds.

Chapter 4

Multiplicative Structures

Now that we have a good grasp of MU, we want to find some of its properties. It occurs that MU is closely linked to notions of orientation. To study it in details, we will work on ring spectrum and on complex orientation. After this, we will use all of our previous work to compute cohomology of MU.

4.1 Ring Spectrum

4.1.1 Ring spectrum and multiplicative cohomology

We have seen in the chapter 2 that spectra and cohomology where equivalent notions (up to homotopy). We know that we can give some cohomology (like $H^*(-, R)$ with R a ring) a multiplicative structure. Then, a good question is what structures on spectra can be used to represent this multiplicity. An answer is the notion of ring spectrum, but before this, we need to define the notion of smash product on **Sp**.

Definition 4.1.1 (Smash product of spectra)**.**
Let $E, F \in Ob(\mathbf{Sp})$. We define the ***smash product of spectra*** $E \wedge F$ *as*

$$(E \wedge F)_{2n} = E_n \wedge F_n$$
$$(E \wedge F)_{2n+1} = E_n \wedge F_n \wedge S^1$$

$$p_{2n} = id_{E_n \wedge F_n \wedge S^1}$$
$$p_{2n+1} = p_n^E \wedge p_n^F$$

Smash product of spectra has many good properties.

Proposition 4.1.2.
Let $E, F \in Ob(\mathbf{Sp})$:

1. *Subspectra of $E \wedge F$ of the form $\overline{E} \wedge \overline{F}$, with $\overline{E}$ cofinal in E and $\overline{F}$ cofinal in F are cofinal in $E \wedge F$.*
2. *$E \wedge F$ is isomorphic to $\widetilde{E \wedge F}$:*
$$\widetilde{E \wedge F}_{2n} = E_n \wedge F_n$$
$$\widetilde{E \wedge F}_{2n+1} = E_{n+1} \wedge F_n$$
3. *Let $X \in Ob(\mathbf{CW})$. Then*
$$(E \wedge F) \wedge X = (E \wedge X) \wedge F = E \wedge (F \wedge X)$$
4. *$\forall m \in \mathbb{Z}$*
$$\Sigma^m(E \wedge F) \sim_{Hom} \Sigma^i E \wedge \Sigma^j F \text{ with } i + j = m$$
5. *$\forall E \in Ob(\mathbf{Sp})$*
$$E \wedge \mathbb{S} \cong E$$
6. *Smash product is associative but* ***only*** *commutative up to isomorphism*[1].

Proof of Proposition 4.1.2:

1. Let $\overline{E} \wedge \overline{F}$ be a subspectrum of $E \wedge F$. Let $e \in E_n \wedge F_n$. Due to the structure of smash product on CW complexes 1.4.6, we get that $e = e_1 \wedge e_2$ with $e_1 \in E_n, e_2 \in F_n$. Then, $\exists m$ such that
$$\Sigma'^{2m} e = \Sigma'^m e_1 \wedge \Sigma'^m e_2 \in \overline{E_{n+m}} \wedge \overline{F_{n+m}} = (\overline{E \wedge F})_{2(n+m)}.$$
2. To see that $E \wedge F \cong \widetilde{E \wedge F}$, we consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & E_n \wedge F_n & \xrightarrow{id} & E_n \wedge F_n \wedge S^1 & \xrightarrow{p_n^E \wedge p_n^F} & E_{n+1} \wedge F_{n+1} & \longrightarrow \cdots \\
& \downarrow id & & \downarrow p_n^E \wedge id & & \downarrow id & \\
\cdots \longrightarrow & E_n \wedge F_n & \xrightarrow{p_n^E \wedge id} & E_{n+1} \wedge F_n & \xrightarrow{id \wedge p_n^F} & E_{n+1} \wedge F_{n+1} & \longrightarrow \cdots
\end{array}$$

[1] In fact, the question of a good (that would be associative) smash product on spectra is a very rich one. We here only give and work on a naive definition but further work on the topic has been done and can be found in [AA74], part IV and in [Swi17], chapter 13.

This imply that we can define the spectral function

$$\mathbf{p} : E \wedge F \hookrightarrow \widetilde{E \wedge F}$$

$$\mathbf{p}_{2n} = id, \ \mathbf{p}_{2n+1} = p_n^E \wedge id_{F_n}$$

This induce that $E \wedge F \cong \underline{E \wedge F}$[2] with

$$\underline{E \wedge F}_{2n} = E_n \wedge F_n$$

$$\underline{E \wedge F}_{2n+1} = \Sigma' E_n \wedge F_n$$

Then, we see that $\underline{E \wedge F}$ is cofinal in $\widetilde{E \wedge F}$. So they are isomorphic.

3. By definition.

4. Consider $i, j \in \mathbb{N}$, $i + j = m$. Then,

$$\Sigma^i E \wedge \Sigma^j F \sim_{Hom} E \wedge F \wedge S^i \wedge S^j \cong E \wedge F \wedge S^m \sim_{Hom} \Sigma^m (E \wedge F)$$

Now, fix $i < 0$, $j \geqslant 0$ we have

$$\Sigma^{-i}(\Sigma^i E \wedge \Sigma^j F) \sim_{Hom} \Sigma^i E \wedge \Sigma^j F \wedge S^{-i} \sim_{Hom} E \wedge \Sigma^j F \sim_{Hom} \Sigma^j (E \wedge F)$$

Using the fact that Σ is a natural bijection on **HoSp**, we get that

$$\Sigma^i E \wedge \Sigma^j F \sim_{Hom} \Sigma^{i+j}(E \wedge F)$$

If both i and j are negative, we have

$$\Sigma^{-i-j}(\Sigma^i E \wedge \Sigma^j F) \sim_{Hom} \Sigma^i E \wedge \Sigma^j F \wedge S^{-i} \wedge S^{-j} \sim_{Hom} E \wedge F$$

Thus, similarly to before, $\Sigma^i E \wedge \Sigma^j F \cong \Sigma^{i+j}(E \wedge F)$.

5. Consider the following diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & E_n \wedge S^n & \xrightarrow{\ id\ } & E_n \wedge S^{n+1} & \xrightarrow{p_n^E \Sigma \wedge id} & E_{n+1} \wedge S^{n+1} & \longrightarrow \cdots \\
& \downarrow \mathbf{p}_n & & \downarrow p_{2n} \circ \Sigma \mathbf{p}_n & & \downarrow \mathbf{p}_{n+1} & \\
\cdots \longrightarrow & E_{2n} & \xrightarrow{p_{2n}\Sigma} & E_{2n+1} & \xrightarrow{p_{2n+1}\Sigma} & E_{2(n+1)} & \longrightarrow \cdots
\end{array}$$

with $\mathbf{p}_n = p_{2n-1} \circ \Sigma p_{2(n-1)} \circ \cdots \circ \Sigma^{n-1} p_n$

This gives us an isomorphism between $E \wedge \mathbb{S}$ and $\Sigma' E$ with

$$(\Sigma' E)_{n+1} = \Sigma' E_n, n \geqslant 0$$

This is a cofinal subspectrum of E. Hence, we have that $E \wedge \mathbb{S} \cong E$.

[2] Cellular maps are by definition open and thus have, when bijective, continuous inverses

6. Commutativity is by definition. Associativity is can be seen by

$$(E \wedge F) \wedge G \cong (E \wedge F) \wedge (G \wedge \mathbb{S}) = (F \wedge G) \wedge (E \wedge \mathbb{S}) \cong E \wedge (F \wedge G)$$

□ 4.1.2

Example 4.1.3.

- *Let U be as in 2.1.2, Then, $U \wedge E$ is of the form*

$$(U \wedge E)_{2n} = \bigvee_{i=0}^{\infty} \Sigma^i E_n$$

$$(U \wedge E)_{2n+1} = \bigvee_{i=1}^{\infty} \Sigma^i E_n$$

- *From previous example, we get*

$$(U \wedge U)_{2n} = \bigvee_{i=0}^{\infty} \bigvee_{j=0}^{i+1} S^i$$

$$(U \wedge U)_{2n+1} = \bigvee_{i=0}^{\infty} \bigvee_{j=0}^{i} S^i$$

With our smash product for spectra, we muss also define smash product for spectral maps.

Definition 4.1.4 (Smash product of spectral maps)**.**
Let $E, F, H, K \in Ob(\mathbf{Sp})$, $f = [\bar{E}, f] \in \mathbf{Sp}(E, H), g = [\bar{F}, g] \in \mathbf{Sp}(F, K)$. We define the ***smash product of f and g****:*

$$f \wedge g : E \wedge F \to H \wedge K$$
$$[\overline{E} \wedge \overline{F}, h]$$

h given as follows.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \overline{E}_n \wedge \overline{F}_n & \xrightarrow{\iota} & \overline{E}_n \wedge \overline{F}_n \wedge S^1 & \xrightarrow{p_n^E \wedge p_n^F} & \overline{E}_{n+1} \wedge \overline{F}_{n+1} & \longrightarrow \cdots \\
& \downarrow f_n \wedge g_n & & \downarrow f_n \wedge g_n \wedge id_{S^1} & & \downarrow f_{n+1} \wedge g_{n+1} & \\
\cdots \longrightarrow & H_n \wedge K_n & \xrightarrow{\iota} & H_n \wedge K_n \wedge S^1 & \xrightarrow{p_n^H \wedge p_n^K} & H_{n+1} \wedge K_{n+1} & \longrightarrow \cdots
\end{array}$$

Unsurprisingly, map create using the smash product behave well under homotopy.

Proposition 4.1.5. *Let $f, g \in \mathbf{Sp}(E, H)$, $h, k \in \mathbf{Sp}(F, K)$. If $f \sim_{Hom} g$, $h \sim_{Hom} k$. Then*

$$f \wedge h \sim_{Hom} g \wedge k$$

In particular, if $[f] = 0$, then $[f \wedge h] = 0$

Proof of Proposition 4.1.5:
Let $W : E \wedge I^+ \to H$ be the homotopy between f and g. Then,

$$W \wedge h : E \wedge I^+ \wedge F \to H \wedge K$$

Gives us the homotopy between $f \wedge h$ and $g \wedge h$.
Now, because $0 = [*]$ the base point map, we get that $* \wedge h = *$ by definition of smash product.

□ 4.1.5

Using this, we can, like for 2.1.47, extend the notion of homotopy to Sp.

Definition 4.1.6 (Homology of spectra)**.**
*Let $E, R \in Ob(\mathbf{Sp})$. We define the E-**homology of the spectrum** R as*

$$E_k(R) = [\Sigma^k \mathbb{S}, R \wedge E] = \pi_k(R \wedge E) = \ colim_n\ \pi_{k+2n}(E_n \wedge R_n)$$

Note that it can be shown that smash product on spectra preserve cofibre sequence as defined in 2.1.40. Thus, if $G \xrightarrow{f} H \xrightarrow{g} K$ is a cofibre sequence, then so does $E \wedge G \xrightarrow{id_E \wedge f} E \wedge H \xrightarrow{id_E \wedge g} E \wedge K$ and, using 2.1.44,

$$E_k(G) \xrightarrow{(id_E \wedge f)_*} E_k(H) \xrightarrow{(id_E \wedge g)_*} E_k(K)$$

Now that we have a spectral smash product, we can give a proper definition of ring spectra.

Definition 4.1.7 (Ring spectrum)**.**
Let $E \in Ob(\mathbf{Sp})$, $m : E \wedge E \to E$ and $\eta : \mathbb{S} \to E$ such that:

1. *The following diagram commute up to homotopy:*

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{id_E \wedge m} & E \wedge E \\ \downarrow{\scriptstyle m \wedge id_E} & & \downarrow{\scriptstyle m} \\ E \wedge E & \xrightarrow{m} & E \end{array}$$

2. *The following diagram commute up to homotopy:*

$$\begin{array}{ccccc} \mathbb{S} \wedge E & \xrightarrow{\eta \wedge id_E} & E \wedge E & \xleftarrow{id_E \wedge \eta} & E \wedge \mathbb{S} \\ & \searrow{\scriptstyle =} & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle =} & \\ & & E & & \end{array}$$

We name (E, m, η) a ***ring spectrum****.*[3]
Furthermore, if $m \sim_{Hom} m \circ T$ with T the map that switch element, then we say that (E, m, η) is a ***commutative ring spectrum****.*

Example 4.1.8.

1. *Let $E = \mathbb{S}$, $m : \mathbb{S} \wedge \mathbb{S} \to \mathbb{S}$ be induced by $m_{i,j} : S^i \wedge S^j \cong S^{i+j}$ and $\eta = id_{\mathbb{S}}$. Then,*

$$(\mathbb{S}, m, \eta) \text{ is a commutative ring spectrum.}$$

2. *Consider the following maps*

$$m_{i,j} : G^{\mathbb{C}}_{i,k_1} \times G^{\mathbb{C}}_{j,k_2} \hookrightarrow G^{\mathbb{C}}_{i+j,k_1+k_2}$$

$$m_{i,j}(x, y) = x \oplus y$$

[3] We see that those diagram are the same that for ring in **Ab**.

Thoses maps induced the following structure on the tautological bundle

$$m_{i,j}^*(\gamma_{i+j,k_1+k_2}) = \gamma_{i,k_1} \times \gamma_{j,k_2}$$

This extend to the universal bundle

$$m_{i,j}^*(\gamma_{i+j}) = \gamma_i \times \gamma_j$$

$$\textit{Thus, we get } T(m_{i,j}) : MU(i) \wedge MU(j) \hookrightarrow MU(i+j)$$

Furthermore, we have that $m_{i,j}$ *is stable with the suspensions maps of* MU*. It therefore extend to a spectral map*

$$m : MU \wedge MU \to MU$$

For η*, because* $BU(0) = *, MU(0) = S^0$*, consider*

$$\eta : \mathbb{S} \to MU$$

$$\eta = [\mathbb{S}, \Sigma^\infty id_{S_0}]$$

$$(MU, m, \eta) \textit{ is a commutative ring spectrum.}$$

Now that we know ring spectrum, let's define multiplicative cohomology and see the link between thoses two notions.

Definition 4.1.9 (Multiplicative cohomology)**.**
Let $\{E^*, \sigma^*\}$ *be a reduced cohomology. We say that it is a* ***multiplication cohomology*** *if there exists a natural transformation for all* $i, j \in \mathbb{Z}$

$$\mu_{i,j}(X, Y) : E^i(X) \otimes E^j(Y) \to E^{i+j}(X \wedge Y)$$

$$1 \in E^0(S^0)$$

satisfying the following axioms:

1. $\mu_{i,j}(\sigma^{i+1} \otimes id) = \sigma^{i+j+1} \circ \mu_{i+1,j}$

$$\begin{array}{ccc} E^{i+1}(\Sigma X) \otimes E^j(Y) & \xrightarrow{\mu_{i+1,j}} & E^{i+j+1}(\Sigma X \wedge Y) \\ \Big\Downarrow \sigma^{i+1} \otimes id & & \Big\Downarrow \sigma^{i+j+1} \\ E^i(X) \otimes E^j(Y) & \xrightarrow{\mu_{i,j}} & E^{i+j}(X \wedge Y) \end{array}$$

2. $\mu_{i,j}(id \otimes \sigma^{j+1}) = \sigma^{i+j+1} \circ \mu_{i,j+1}$

$$\begin{array}{ccc} E^i(X) \otimes E^{j+1}(\Sigma Y) & \xrightarrow{\mu_{i,j+1}} & E^{i+j+1}(X \wedge \Sigma Y) \\ \Big\Downarrow id \otimes \sigma^{j+1} & & \Big\Downarrow \sigma^{i+j+1} \\ E^i(X) \otimes E^j(Y) & \xrightarrow{\mu_{i,j}} & E^{i+j}(X \wedge Y) \end{array}$$

3. $\mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu)$

4. $\mu(1 \otimes x) = \mu(x \otimes 1) = x, \forall x \in E^n(X)$

Furthermore, if μ is commutative, we say that $H^*(-)$ is a commutative cohomology.

Definition 4.1.10 (Cohomology ring)**.**
Let $\{E^*\}$ be a multiplicative cohomology. Consider

$$E^\bullet(X) = \bigoplus_{i \in \mathbb{Z}} E^i(X)$$

Equipped with ***cup product*** *induced by multiplicative structure and diagonal map:*

$$\smile : E^i(X) \otimes E^j(X) \xrightarrow{\mu_{i,j}} E^{i+j}(X \wedge X) \xrightarrow{\Delta^*} E^{i+j}(X)$$

We get that $(H^\bullet(X), +, *, \smile, 1)$ is a graded ring. We name $E^\bullet(X)$ the ***cohomology ring*** *of X.*

Theorem 4.1.11.
Let (E, m, η) be a ring spectrum.This induce on the cohomology $E^*(-)$ a multiplicative structure.

Proof of Theorem 4.1.11:
We construct our multiplication using the fact that $E^n(X) = [\Sigma^\infty X, \Sigma^n E]$, we consider $f \in E^i(X)$, $g \in E^j(Y)$.

$$\mu_{i,j}(X,Y) : [\Sigma^\infty X, \Sigma^i E] \otimes [\Sigma^\infty Y, \Sigma^j E] \to [\Sigma^\infty (X \wedge Y), \Sigma^{i+j} E]$$

$$\mu_{i,j}(X,Y)(f,g) = \Sigma^{i+j} m(f \wedge g).$$

Using $\Sigma^{i+j} m : \Sigma^i E \wedge \Sigma^j E \sim_{Hom} \Sigma^{i+j}(E \wedge E) \to \Sigma^{i+j} E$.

We also define $1 = [\eta] \in E^0(S^0) = [\mathbb{S}, E]$.
To understand why μ satisfy the first two point, we see using 2.1.45, that $\sigma^i(f) = \Sigma^{-1} f$. Then

$$\begin{array}{rcl} \mu_{i,j} \circ (\sigma^{i+1} \otimes id)(f \otimes g) & = & \Sigma^{i+j} m(\Sigma^{-1} f \wedge g) \\ & = & \Sigma^{i+j} m\big(\Sigma^{-1}(f \wedge g)\big) \\ & = & \Sigma^{-1}\Sigma^{i+j+1} m\big(f \wedge g\big) \\ & = & \sigma^{i+j+1} \circ \mu_{i+1,j}(f \otimes g) \end{array}$$

This goes similarly in the other case.
Now, the following diagram commute up to homotopy

$$\begin{array}{ccc} \Sigma^i E \wedge \Sigma^j E \wedge \Sigma^k E & \xrightarrow{id_{\Sigma^i E} \wedge \Sigma^{j+k} m} & \Sigma^i E \wedge \Sigma^{j+k} E \\ \downarrow{\scriptstyle \Sigma^{i+j} m \wedge id_{\Sigma^k E}} & & \downarrow{\scriptstyle \Sigma^{i+j+k} m} \\ \Sigma^{i+j} E \wedge \Sigma^k E & \xrightarrow{\Sigma^{i+j+k} m} & \Sigma^{i+j+k} E \end{array}$$

Therefore,

$$\begin{array}{rcl} \mu(\mu \otimes id_{\Sigma^k E})(f, g, h) & = & \mu\big(\Sigma^{i+j} m(f \wedge g), h\big) \\ & = & \Sigma^{i+j+k} m\big(\Sigma^{i+j} m(f \wedge g) \wedge h\big) \\ & \sim_{Hom} & \Sigma^{i+j+k} m\big(f \wedge \Sigma^{j+k} m(g \wedge h)\big) \\ & = & \mu\big(f \wedge \Sigma^{j+k} m(g \wedge h)\big) \\ & = & \mu(id_{\Sigma^i E} \otimes \mu)(f, g, h) \end{array}$$

Giving us our first equality. The second equality comes from

$$\mu(1 \otimes x) = m(\eta \wedge id_{\Sigma^j E})(x) = x = m(id_{\Sigma^i E} \wedge \eta)(x) = \mu(x \otimes 1)$$

Furthermore, we see that if E is a commutative ring spectrum, we get that μ is commutative.
□ 4.1.11

Let's now try to prove that multiplicative cohomology → ring spectrum. Sadly, it isn't as easy and we can only show here this weaker result.

Theorem 4.1.12.
Let $(E^*(-), \mu, 1)$ be a multiplicative cohomology. Furthermore, let

$$\lim_n^1 E^{n-1}(E_n) = \lim_{2n}^1 E^{2n-1}(E_n \wedge E_n) = \lim_{4n}^1 E^{4n-1}\big(E_{2n} \wedge (E_n \wedge E_n)\big) = 0$$

with $\lim^1$ as defined here.
Then, we can give the spectrum E a ring spectrum structure.

Proof of Theorem 4.1.12:
wlog, we may assume that $E \in Ob(\mathbf{\Omega Sp})$ with trivial injections. Using

$$\mu_{i,j}(X,Y) : [X, E_i] \otimes [Y, E_j] \to [X \wedge Y, E_{i+j}]$$

We define $\eta = [\Sigma^\infty 1]$. For $m : E \wedge E \to E$, we use

$$\overline{m_{2n}} = \mu_{n,n}\Big([id_{E_n}][id_{E_n}]\Big)$$

$$\overline{m_{2n+1}} = \Sigma m_{2n}$$

We have to see if those maps agree with each others.
σ^n is given by the following diagram:

$$\begin{array}{ccc}
[X, E_n] & \xrightarrow{\quad\Sigma\quad} & [\Sigma X, \Sigma E_n] \\
\Big\| (p'_{n+k})_* & \overset{\sigma^{n+1}}{\nwarrow} & \Big\downarrow (p_{n+k})_* \\
[X, \Omega E_{n+1}] & \overset{A}{=\!=\!=\!=} & [\Sigma X, E_{n+1}]
\end{array}$$

and we have this other diagram

$$\begin{array}{ccc}
[E_{n+1}, E_{n+1}] \otimes [E_{n+1}, E_{n+1}] & \xrightarrow{\mu_{n+1,n+1}} & [E_{n+1} \wedge E_{n+1}, E_{2(n+1)}] \\
\iota^* \otimes \iota^* \Big\downarrow & & \Big\downarrow \iota^* \\
[\Sigma E_n, E_{n+1}] \otimes [\Sigma E_n, E_{n+1}] & \xrightarrow{\mu_{n+1,n+1}} & [\Sigma^2(E_n \wedge E_n), E_{2(n+1)}] \\
\sigma^{n+1} \otimes \sigma^{n+1} \Big\Downarrow & & \Big\Downarrow \sigma^{2n+1} \circ \sigma^{2(n+1)} \\
[E_n, E_n] \otimes [E_n, E_n] & \xrightarrow{\mu_{n,n}} & [E_n \wedge E_n, E_{2n}]
\end{array}$$

This gives us, seeing $[\overline{m_{2(n+1)}}|_{\Sigma E_n \wedge \Sigma E_n}] = [\mu_{n+1,n+1}(\Sigma id_{E_n} \oplus \Sigma id_{E_n})]$ that

$$\overline{m_{2(n+1)}}|_{\Sigma E_n \wedge \Sigma E_n} \sim_{Hom} \Sigma^2 \overline{m_{2n}}$$

Thus, similarly to what we did in the proof of 2.2.18 and in 2.2.17, we can transform $\{\overline{m_n}\}$ into a spectral map.

$$m : E \wedge E \to E$$

Furthermore, using 2.2.17, we have that m is unique up to homotopy. [4]
Now, we have to show that (E, m, η) is a ring spectrum. This comes from the fact that

[4] Meaning that if $g : E \wedge E \to E$ with $g_n \sim_{Hom} \overline{m_n}$ for $n \in \mathbb{N}$, then $g \sim_{Hom} m$.

$$\begin{aligned}
\Big(m(id_E \wedge m)\Big)_{4n} &\sim_{Hom} \mu_{2n,2n}\Big(id_{E_{2n}} \otimes \mu_{n,n}(id_{E_n} \otimes id_{E_n})\Big)\\
&\sim_{Hom} \mu_{2n,2n}\Big(\mu_{n,n}(id_{E_n} \otimes id_{E_n}) \otimes id_{E_{2n}}\Big)\\
&\sim_{Hom} (m(m \wedge id_E))_{4n}
\end{aligned}$$

And thus, using $\Big(m(id_E \wedge m)\Big)_{4n+i} = \Sigma^i \Big(m(id_E \wedge m)\Big)_{4n}$ with $0 \leqslant i \leqslant 3$, we get that $\forall$n$\in \mathbb{Z}$

$$\Big(m(id_E \wedge m)\Big)_n \sim_{Hom} \Big(m(m \wedge id_E)\Big)_n$$

Now, using cofinality, we can forget everything bellow 0 in $E \wedge E \wedge E$ and use 2.2.17. Because $\lim^1_{4n} E^{4n-1}\big(E_{2n} \wedge (E_n \wedge E_n)\big) = 0$, we get that

$$m(id_E \wedge m) \sim_{Hom} m(m \wedge id_E)$$

Similarly,

$$\begin{aligned}
m(id_E \wedge \eta)_{2n} &\sim_{Hom} \mu_{n,n}(id_{E_n} \otimes 1)\\
&\sim_{Hom} id_{E_{2n}}\\
&\sim_{Hom} m(\eta \wedge id_E)_{2n}
\end{aligned}$$

$$\text{Thus,} \forall n \in \mathbb{Z}, m(id_E \wedge \eta)_n \sim_{Hom} id_{E_n} \sim_{Hom} m(id_E \wedge \eta)_n$$

Therefore, because $\lim^1_n E^{2n-1}(E_n \wedge S^n) = \lim^1_n E^{n-1}(E_n) = 0$, using 2.2.17, we get that

$$m(\eta \wedge id_E) \sim_{Hom} id_E \sim_{Hom} m(id_E \wedge \eta).$$

Similarly, if μ is commutative, then $(m \circ T)_n \sim_{Hom} m$ and by using 2.2.17, $m \circ T \sim_{Hom} m$, meaning that E is a commutative.

□ 4.1.11

From this proof, it appears that Brown's representation on multiplicative cohomology is far harder than what it appear at first glance and is in fact a very deep question. Further work on this subject can be fund in [AA74].

4.1.2 Thom isomorphism and orientation on ring spectrum

Now that we have a good grasp of ring spectrum and thus of multiplicative structure, we will study a very important property of MU spectrum, the complex orientation 4.1.23, a property that simplifies computation of cohomology via what is called Thom-Dolt isomorphism 4.1.15 and gives further structure to spectra.

Definition 4.1.13 (Thom class)**.**
Let ξ be a complex n dimensional vector bundle on $X \in Ob(\mathbf{CW})$ and E be a ring spectrum. For every $b \in X$, the inclusion map $\iota_b : b \hookrightarrow X$ induce a map

$$T(\iota_b) : S^{2n} = T(\epsilon^n_{\mathbb{C}}(b)) \to T(\xi).$$

*We name a **Thom class** an element $u \in E^{2n}\big(T(\xi)\big)$ such that $\forall b \in X$, $T(\iota_b)^*(u)$ is a generator of $E^{2n}(S^{2n})$. That is to say*

$$T(\iota_x)^*(u) = \epsilon\delta_n$$

with $\epsilon \in E^0(S^0)$ is unit, seeing $(E^0(S^0), \mu, 1)$ as a ring.
$\delta_n = \sigma(1) \in E^n(S^n)$ using $\sigma : E^0(S^0) \cong E^n(S^n)$.

*If such a Thom class exists, we say that ξ is E-**orientable.*** [5]

A good question is how orientable vector behave with standard tools of vectors bundles.

Proposition 4.1.14.

- *Let E be a ring module and ξ be a E-orientable n dimensional vector bundle on $Y \in Ob(\mathbf{CW})$ with Thom class u. Let $f \in \mathbf{CW}(X, Y)$. Then, $f^*(\xi)$ is E-orientable.*
- *Let E be a ring module, ξ_1 be a E-orientable n_1 dimensional vector bundle on X with Thom class u_1 and ξ_2 be a E-orientable n_2 dimensional vector bundle on Y with Thom class u_2. Then, $\xi_1 \times \xi_2$ is also E-orientable.*

[5] We name it orientable because this can in a way be seen as an extension of how we define orientation on manifold using tangent bundle [Gre18] 22.1.

Proof of Proposition 4.1.14:

- Using the Thomification $T(f) : T\big(f^*(\xi)\big) \to T(\xi)$. This gives us a map

$$E^{2n}(T(\xi)) \xrightarrow{T(f)^*} E^{2n}\big(T(f^*(\xi))\big)$$

Consider $T(f)^*(u)$. We will show that it is a Thom class. To do so, we need the following commutative diagram:

$$\begin{array}{ccc} & & T\big(f^*(\xi)\big) \\ & \nearrow^{T(\iota_b)} & \Big\downarrow T(f) \\ S^{2n}_{\mathbb{R}} & \xrightarrow{T(\iota_{f(b)})} & T(\xi) \end{array}$$

Thus, $\forall b \in X$

$$T(\iota_b)^* \circ T(f)^*(u) = \big(T(f \circ \iota_b)\big)^*(u) = T(\iota_{f(b)})^*(u) = \epsilon\delta_n.$$

- Using the fact that $T(\xi_1 \times \xi_2) = T(\xi_1) \wedge T(\xi_2)$, we can consider the multiplication

$$\mu : E^{2n_1}(T(\xi_1)) \otimes E^{2n_2}(T(\xi_2)) \to E^{2(n_1+n_2)}(T(\xi_1) \wedge T(\xi_2))$$

Consider $u = \mu(u_1, u_2)$. Let's show this is a Thom class. We see that $\forall a \in X \times Y$, $a = (b, b')$. We thus have the following diagram.

$$\begin{array}{ccc} S^{2(n_1+n_2)} & \xrightarrow{T(\iota_a)} & T(\xi_1 \times \xi_2) \\ \Big\| & & \Big\| \\ S^{2n_1} \wedge S^{2n_2} & \xrightarrow{T(\iota_b) \wedge T(\iota_{b'})} & T(\xi_1) \wedge T(\xi_2) \end{array}$$

Therefore,

$$\begin{aligned} T(\iota_a)^* u &= \big(T(\iota_b), T(\iota_{b'})\big)^* \mu(u_1, u_2) \\ &= \mu(T(\iota_b)^* u_1, T(\iota_{b'})^* u_2) \\ &= \mu(\epsilon_1 \delta_{n_1}, \epsilon_2 \delta_{n_2}) \\ &= \epsilon \delta_{n_1+n_2} \end{aligned}$$

□ 4.1.14

Now, let's show a very powerful theorem. It can be seen as a generalisation of the suspension rule for cohomology.

Theorem 4.1.15 (Thom-Dold isomorphism)**.**
Let E be a ring module and ξ be a E-orientable n dimensional vector bundle on $X \in Ob(\mathbf{CW})$ with Thom class u. Then, using the following equivalences and functions

- $D(\xi) \sim_{Hom} X$
- $T(\xi) = D(\xi)/S(\xi) \sim_{Hom} D(\xi) \cup C\big(S(\xi)\big)$
- $\Delta : X \cup CA \to X \wedge (X \cup CA)$

we construct $\forall k \in \mathbb{Z}$ the morphism:

$$\Psi : E^k(X^+) = E^k\big(D(\xi)^+\big) \xrightarrow{\mu(-,u)} E^{k+2n}\big(D(\xi)^+ \wedge T(\xi)\big) \xrightarrow{\Delta^*} E^{k+2n}(T(\xi))$$

*We name Ψ the **Thom-Dolt isomorphism** and as its name induce, it is an isomorphism*

$$\Psi : E^k(X^+) \cong E^{k+2n}(T(\xi)).^6$$

$$[a] \to [a \smile u]$$

Proof of Theorem 4.1.15:

1. First, suppose that $\xi = \epsilon^n_{\mathbb{C}}(X)$. Then, $T(\xi) = X^+ \wedge S^{2n}_{\mathbb{R}}$ and Ψ is given by

$$\Psi : E^k(X) \to E^{k+2n}(X^+ \wedge S^{2n}_{\mathbb{R}})$$

$$\Psi(f) = \mu(f, T(\iota)^* u)$$

But this is given by the following diagram:

$$\begin{array}{ccc} E^k(X^+) & \xrightarrow{\quad\Psi\quad} & E^{k+2n}(X^+ \wedge S^{2n}_{\mathbb{R}}) \\ \Big\| \mu(-,1) & & \mu(-,\epsilon) \Big\| \\ E^k(X^+ \wedge S^0_{\mathbb{R}}) & \overset{id_{X^+} \wedge \sigma}{=\!=\!=\!=} & E^{k+2n}(X^+ \wedge S^{2n}_{\mathbb{R}}) \end{array}$$

Thus, Ψ is an isomorphism.

[6] It is usually found in literature using unreduced cohomology on left and reduced cohomology in the right.

2. Now, let A, B be open subsets of X, $\xi_A = \iota_A^*(\xi), \xi_B = \iota_B^*(\xi), \xi_\cap = \iota_{A\cap B}^*(\xi), \xi_\cup = \iota_{A\cup B}^*(\xi)$. They all have Thom space and class. We now suppose that the Thom isomorphism works for ξ_A, ξ_B and $\xi_\cap$. Using the following cofibre sequence

$$(A\cap B)^+ \xrightarrow{\iota} A^+ \vee B^+ \xrightarrow{(id_A, id_B)} (A\cup B)^+$$

we get, using 2.1.41 the Mayer-Vietoris long exact sequence for cohomology:

$$E^k\big((A\cap B)^+\big) \leftarrow E^k(A^+)\oplus E^k(B^+) \leftarrow E^k\big((A\cup B)^+\big) \leftarrow E^{k-1}\big((A\cap B)^+\big) \leftarrow E^{k-1}(A^+)\oplus E^{k-1}(B^+)$$

It works similarly for Thom space, that form a cofibre sequence

$$T(\xi_\cap) \to T(\xi_A) \vee T(\xi_B) \to T(\xi_\cup)$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccccccc}
E^k((A\cap B)^+) & \longleftarrow & E^k(A^+)\oplus E^k(B^+) & \longleftarrow & E^k((A\cup B)^+) & \longleftarrow & \cdots \\
\Big\Downarrow \Psi & & \Big\Downarrow \Psi\oplus\Psi & & \Big\downarrow \Psi & & \\
E^{k+2n}(T(\xi_\cap)) & \longleftarrow & E^{k+2n}(T(\xi_A))\oplus E^{k+2n}(T(\xi_B)) & \longleftarrow & E^{k+2n}(T(\xi_\cup)) & \longleftarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots \leftarrow & E^k((A\cup B)^+) & \longleftarrow & E^{k-1}((A\cap B)^+) & \longleftarrow & E^{k-1}(A^+)\oplus E^{k-1}(B^+) \\
& \Big\downarrow \Psi & & \Big\Downarrow \Psi & & \Big\Downarrow \Psi\oplus\Psi \\
\cdots \leftarrow & E^{k+2n}(T(\xi_\cup)) & \longleftarrow & E^{k-1+2n}(T(\xi_\cap)) & \longleftarrow & E^{k-1+2n}(T(\xi_A))\oplus E^{k-1+2n}(T(\xi_B))
\end{array}$$

Using 5-Lemma, we get that

$$\Psi : E^k(A\cup B) \cong E^{k+2n}(T(\xi_\cup))$$

3. Lets now use the previous 2 points with the trivial covering $\{U_\alpha\}$ of X. If there are only finitely many such open, our proof is done. Issues occurs if they are infinitely many. To solve this problem, because X is a paracompact, we can assume that $\{U_\alpha\}$ is countable. Then consider the open $V_n = \cup_{i=0}^n U_n$, with $\xi_{V_n} = p^{-1}(V_n)$. Using the previous 2 points, $\{V_n\}$ follows the Thom-Dolt isomorphism.

   Because $\bigcup_{n\in\mathbb{N}} V_n = X$, using 1.4.21, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim^1_m E^k(V_m) & \longrightarrow & E^k(X) & \longrightarrow & \lim_m E^k(V_m) & \longrightarrow & 0 \\
 & & \Big\Downarrow \lim^1 \Psi & & \Big\downarrow \Psi & & \Big\Downarrow \mathrm{colim}\Psi & & \\
0 & \longrightarrow & \lim^1_m E^{k+2n}(T(\xi_{V_m})) & \longrightarrow & E^k(T(\xi)) & \longrightarrow & \lim_m E^{k+2n}(T(\xi_{V_m})) & \longrightarrow & 0
\end{array}$$

Thus, using 5-Lemma once again, we get our desired result

$$\Psi : E^k(X^+) \cong E^{k+2n}(T(\xi)).$$

□ 4.1.15

Corollary 4.1.16.

Let $X \in Ob(\mathbf{CW})$, E be a ring spectrum and let ξ be a E-orientable n complex vector bundle on X. Then

$$E^{k+2n}(E^{\dagger}_{\xi}) \cong E^k(X^+)$$

with $E^{\dagger}_{\xi}$ the one point compactification on the total space of ξ.

Proof of Corollary 4.1.16: Using Thom-Dolt isomorphism 4.1.15 and 3.1.26. □ 4.1.16

Now, the good question to ask ourselves is when do we have Thom class ? We can give a beginning of answer using the following proposition.

Proposition 4.1.17.
Let E be a ring spectrum. If γ_n is E-orientable, then any n complex vector bundle is E-orientable.

Proof of Proposition 4.1.17:
Using the fact that γ_n is the universal bundle 3.1.34, any n complex bundle ξ on X is given by $\xi = f^*(\gamma_n)$. Using 4.1.14, we get that ξ is E-oriented.

□ 4.1.17

Thus, the question of whether or not bundles are oriented is not as much a question about bundle but is a question about E itself. To continue, we will need to define the notion of complex orientation on ring spectrum.

Definition 4.1.18 (Complex oriented ring spectrum)**.**
*Let E be a ring spectrum. We say that it is **complex oriented** if:*

1. *Every ξ complex vector bundle has a Thom class u_ξ.*
2. *$u_{f^*(\xi)} = T(f)^* u$, given by 4.1.14.*
3. *$u_{\xi_1 \times \xi_2} = \mu(u_{\xi_1}, u_{\xi_2})$ given by 4.1.14.*

*We name $\{u_\xi\}$ a **complex orientation on** E.*

Using universal bundles, we can reduce theses conditions.

Lemma 4.1.19.
Let E be a ring spectrum. It is complex oriented if and only if γ_n are E-oriented using $u_n \in E^{2n}(MU(n))$ such that $\forall i, j \in \mathbb{N}$:

$$\mu(u_i, u_j) = T(m_{i,j})^*(u_{i+j})$$

with $T(m_{i,j}) : MU(i) \wedge MU(j) \to MU(i+j)$.

Proof of Lemma 4.1.19:

- $\underline{\Rightarrow}$: By definition.
- $\underline{\Leftarrow}$: Because γ_n are universal bundles using 3.1.34, any n complex vector bundle is of the form $\xi = f^*(\gamma_n)$. Thus, we can construct using 4.1.14 a Thom class on ξ, $u_\xi = T(f)^* u_n$.

  Now, we have to check that the latter points of definition 4.1.18 are preserved.

  - $u_{f^*(\xi)} = T(f)^* u$: Let $\xi_1 = g_1^*(\gamma_n)$, $\xi_2 = g_2^*(\gamma_n)$ such that $\xi_1 = f^*(\xi_2)$. Then, $\xi_1 = (g_2 \circ f)^*(\gamma_n)$. Using 3.1.34, we get that $g_1 \sim_{Hom} g_2 \circ f$. Thus,

    $$u_{\xi_1} = T(g_1)^* u_n = T(g_2 \circ f)^* u_n = T(f)^* u_{\xi_2}.$$

  - $u_{\xi_1 \times \xi_2} = \mu(u_{\xi_1}, u_{\xi_2})$: Let $\xi_1 = g_1^*(\gamma_i)$ be a bundle on X, $\xi_2 = g_2^*(\gamma_j)$ be a bundle on Y. We have that

    $$f^*(\gamma_{i+j}) = \xi_1 \times \xi_2 = (g_1 \times g_2)^*(\gamma_i \times \gamma_j)$$

$$\gamma_i \times \gamma_j = m_{i,j}^*(\gamma_{i+j}).$$

Thus, we have that

$$\begin{aligned} u_{\xi_1 \times \xi_2} &= T(f)^*(u_{i+j}) \\ &= T\big(m_{i,j} \circ g_1 \times g_2\big)^*(u_{i+j}) \\ &= T(g_1 \times g_2)^* T(m_{i,j})^*(u_{i+j}) \\ &= T(g_1 \times g_2)^* \mu(u_i, u_j) \\ &= \mu\big(T(g_1)^* u_i, T(g_2)^* u_j\big) \\ &= \mu(u_{\xi_1}, u_{\xi_2}) \end{aligned}$$

□ 4.1.19

Corollary 4.1.20.
The MU spectrum is complex oriented.

Proof of Corollary 4.1.20:
We take as Thom class for γ_n

$$u_n \in MU^{2n}(MU(n)) = [\Sigma^\infty MU(n), \Sigma^{2n} MU]$$

$$u_n = \Sigma^\infty id_{MU(n)}$$

Let's show that u_n is indeed a Thom class. We remind ourselves that $1 = [\eta] = [\Sigma^\infty id_{S^0}]$. Thus, $\delta_n = [\Sigma^\infty id_{S^{2n}}]$. Then, $\forall b \in X$

$$T(\iota_b)^*(u_n) = [\Sigma^\infty id_{MU(n)} \circ \Sigma^\infty T(\iota_b)] = [\Sigma^\infty \big(id_{MU(n)} \circ T(\iota_b)\big)] = [\Sigma^\infty T(\iota_b)] = [\Sigma^\infty id_{S^{2n}}] = \delta_n.$$[7]

We thus have Thom class.
Now, what is left to show is that $\mu(u_i, u_j) = T(m_{i,j})^*(u_{i+j})$. But this comes from the multiplicative structure of MU.

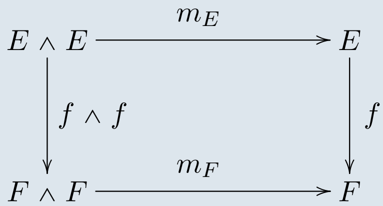
$$\mu(u_i, u_j) = T(m_{i,j})^*(u_i \wedge u_j) = T(m_{i,j})^*(u_{i+j}).$$

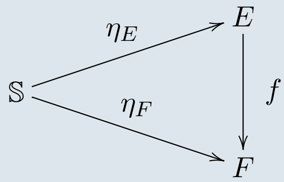
□ 4.1.20

[7]This comes from $T(\iota_b) \sim_{Hom} id_{S^2 n}$ because $BU(n)$ is connected. See 4.2.20 to understand why

Now, let's add some structure to spectral maps, making them like ring morphisms.

Definition 4.1.21 (Ring spectral maps)**.**
Let $(E, m_E, \eta_E), (F, m_F, \eta_F)$ *be ring spectra.* $f \in \mathbf{Sp}(E, F)$ *is a* ***ring spectral map*** *if the following diagrams commute up to homotopy*

- 

- 

It is apparent from the definition that being a ring map is conserved by homotopy.

Lemma 4.1.22.
Let E, F *be ring spectra,* $f : E \to F$ *be a ring map. Let* $\{u_\xi\}$ *be a complex orientation on* E*. Then, using* f*, we can give* F *a complex orientation.*

Proof of Lemma 4.1.22:
Given $\{u_\xi\}$, we define the complex orientation on F as

$$\{v_\xi = f_*(u_\xi)\}$$

We see that v_ξ is Thom class. Indeed, $\forall b \in X$:

$$\begin{array}{rcl} T(\iota_b)^* v_\xi & = & T(\iota_b)^* f_* u_\xi \\ & = & f_* T(\iota_b)^* u_\xi \\ & = & f_* \mu_E(\epsilon, \delta_n^E) \\ & = & \mu_F(f_*\epsilon, f_*\delta_n^E) \\ & = & \mu_F(\epsilon', \delta_n^F) \end{array}$$

To see that this form a complex orientation, using 4.1.19, we simply have to show that

$$\mu_F(v_i, v_j) = T(m_{i,j})^*(v_{i+j}).$$

But this is because

$$\begin{array}{rcl}\mu_F(v_i, v_j) & = & \mu_F(f_* u_i, f_* u_j) \\ & = & f_* \mu_E(u_i, u_j) \\ & = & f_* T(m_{i,j})^*(u_{i+j}) \\ & = & T(m_{i,j})^*(f_* u_{i+j}) \\ & = & T(m_{i,j})^*(v_{i+j})\end{array}$$

□ 4.1.22

The last lemma induce in fact a very important property on MU. It fully define complex orientation.

Theorem 4.1.23 (Universal complex orientation theorem)**.**
Let E be a ring spectrum. There exists a bijection between the followings sets

$$\Big\{ f \in [MU, E] |\ f \text{ a ring map } \Big\} \cong \Big\{ \{u_\xi\} |\ \{u_\xi\} \text{ complex orientation on } E \Big\}$$

In particular, E is complex oriented $\iff \exists f : MU \to E$, f a ring spectral map.

Proof of Theorem 4.1.23:
We have our first side given by 4.1.22. We name the complex orientation given in 4.1.20 as $\{mu_\xi\}$. We get

$$\Big\{ [f] \in \mathbf{HoSp}(MU, E) |\ f \text{ a ring map } \Big\} \to \Big\{ \{u_\xi\} |\ \{u_\xi\} \text{ complex orientation on } E \Big\}$$

$$f \to \{f_* mu_\xi\}$$

To construct our inverse, we need to see MU under another light, namely,

$$MU = \ \mathrm{colim}_n \Sigma^{-2n} \Sigma^\infty MU(n)$$

We thus have, by using 2.1.47, that

$$[MU, E] = E^0(MU) \cong \lim_n E^{2n}(MU(n)).$$

with $T(j) : \Sigma^2 MU(n) \hookrightarrow MU(n+1)$ inducing

$$\Sigma^{-2n} \Sigma^\infty MU(n) = \Sigma^{-2(n+1)} \Sigma^\infty \Sigma^2 MU(n) \xrightarrow{\Sigma^{-2(n+1)} \Sigma^\infty j} \Sigma^{-2(n+1)} \Sigma^\infty MU(n+1)$$

Now consider u_n. We are in the following situation

$$u_n \in E^{2n}(MU(n)) = [\Sigma^\infty MU(n), \Sigma^{2n} E] = [\Sigma^{-2n} \Sigma^\infty MU(n), E]$$

Furthermore, we have that by definition $[T(j)^* u_{n+1}] = [u_{\gamma_n \oplus \epsilon}] = [\Sigma^2 u_n]$. This mean that

$$u_{n+1} \circ \Sigma^{-2(n+1)} \Sigma^\infty T(j) \sim_{Hom} \Sigma^2 u_n$$

Thus, all our maps are compatible with each others up to homotopy. Using homotopy extension for spectra 2.1.36 and by operating similarly to what we did in 2.2.17, we construct

$$u : MU \to E$$

$$u = \lim_n u'_n, \text{ with } u'_n \sim_{Hom} u_n$$

Now, to see that u is a ring map. This comes directly from the fact that

$$m_E(u_i \wedge u_j) = \mu_E(u_i, u_j) = T(m_{i,j})^*(u_{i+j}) = u_{i+j} \circ T(m_{i,j})$$

Thus,

$$m_E(u \wedge u) = \lim_{i,j} m_E(u_i, u_j) = \lim_{i,j} u_{i+j} \circ T(m_{i,j}) = u \circ m_{MU}.$$

$$[\eta_E] = 1_E = [u_0] \Rightarrow u \circ \eta_{MU} = [u_0 \circ \Sigma^\infty id_{S^0}] = [u_0].$$

We therefore get

$$\Big\{[f] \in \mathbf{HoSp}(MU, E) |\ f \text{ a ring map } \Big\} \leftarrow \Big\{\{u_\xi\} |\ \{u_\xi\} \text{ complex orientation on } E\Big\}$$

$$u = \lim_n u'_n \leftarrow \{u_\xi\}.$$

We now have to see that those maps are mutual inverses: To do so, we see that, using 4.1.19, it is sufficient to work on $\{u_n\}$.

$$u_*(mu_n) = [u \circ \Sigma^\infty id_{MU(n)}] = [u_n]$$

$$\lim_n f_*(mu_n) = \lim_n [f \circ \Sigma^{-2n} \Sigma^\infty id_{MU(n)}] = [f \circ \lim_n \Sigma^{-2n} \Sigma^\infty id_{MU(n)}] = [f \circ id_{MU}] = [f].$$

□ 4.1.23

Example 4.1.24.
The following spectra are complex oriented:

- MU.
- HR, *with* R *any ring. (See for example [AA74]).*
- $\mathbb{S}$ *isn't complex oriented, otherwise using* η, *we would have that every ring spectrum is complex oriented.*

4.2 Computation of cohomology of MU

Our goal in this section is to compute the cohomology of MU in complex oriented cohomology, a work harder than it appears at first glance. What we know is that

$$\forall k, \gamma_{0,k} = \{*\} \Rightarrow T(\gamma_{0,k}) = S^0 \text{ using 3.1.26}$$

$$\text{Thus}, MU(0) = \text{ colim}_k T(\gamma_{0,k}) = S^0$$

$$E^i(MU(0)) = \pi_{-i}(E).$$

4.2.1 Computation of $\mathbb{C}P^k$

Proposition 4.2.1.
Let $\gamma_{1,k}$ be the tautological line bundle on $\mathbb{C}P^k$. Then :

$$T(\gamma_{1,k}) \cong \mathbb{C}P^{k+1}$$

Proof of Proposition 4.2.1:
To prove this isomorphism, consider $[e_{k+2}] \in \mathbb{C}P^{k+1}$. We construct the following map:

$$\varphi : \gamma_{1,k} \to \mathbb{C}P^{k+1} \backslash \{e_{k+2}\}$$

$$\varphi([x], v) = [x+ < v, x > e_{k+2}]$$

This map is a well defined. Indeed, if $x = \delta y$, then

$$\varphi([x], v) = [\delta y+ < v, \delta y > e_{k+2}] = [\delta(y+ < v, y > e_{k+2})] = [y+ < v, y > e_{k+2}] = \varphi([y], v).$$

φ is in fact an homeomorphism with inverse continuous map

$$\varphi^{-1} : \mathbb{C}P^{k+1} \backslash \{e_{k+2}\} \to \gamma_{1,k}$$

$$\varphi^{-1}([y]) = ([x], \frac{< y, e_{k+2} >}{< x, x >} x)$$

with $x = y- < y, e_{k+2} > e_{k+2}$.
But now, we have using 3.1.26 that

$$T(\gamma_{1,k}) = \gamma_{1,k}^{\dagger} \cong (\mathbb{C}P^{k+1} \backslash \{e_{k+1}\})^{\dagger} \cong \mathbb{C}P^{k+1}.$$

The later isomorphism comes from the fact that $\mathbb{C}P^{k+1}$ is compact Hausdorff. □ 4.2.1

Corollary 4.2.2.

$$\mathbb{C}P^1 \cong T(\gamma_{1,0}) = T(\epsilon_{\mathbb{C}}(*)) \cong S^2_{\mathbb{R}}$$

We can now compute homotopy of $\mathbb{C}P^k$.

Lemma 4.2.3.
Let E be a complex oriented ring spectrum. Then:

$$E^i(\mathbb{C}P^k) \cong \bigoplus_{j=1}^{k} \pi_{i-2j}(E) \text{ for } E^* \text{ reduced}$$

$$E^i(\mathbb{C}P^k) \cong \bigoplus_{j=0}^{k} \pi_{i-2j}(E) \text{ for } E^* \text{ unreduced}$$

Proof of Lemma 4.2.3:
First, using 1.3.7, we have that for

$$E^i(X^+) \cong E^i(X) \oplus E^i(S^0).$$

Now, let's show our formula by induction using Thom-Dolt isomorphism 4.1.15:

- $k = 1$:
$$E^i(\mathbb{C}P^1) = E^i(S^2) \cong E^{i-2}(S^0) = \pi_{i-2}(E)$$
- $k > 1$.
$$\begin{aligned} E^i(\mathbb{C}P^{k+1}) &\cong E^{i-2}((\mathbb{C}P^k)^+) \\ &\cong E^{i-2}((\mathbb{C}P^k)) \oplus E^{i-2}(S^0) \\ &\cong \textstyle\bigoplus_{j=1}^{k} \pi_{i-2(j+1)}(E) \oplus \pi_{i-2}(E) \\ &\cong \textstyle\bigoplus_{j=1}^{k+1} \pi_{i-2j}(E) \end{aligned}$$

□ 4.2.3

Corollary 4.2.4.
Let $H^*(-, R)$ be the standard unreduced cohomology modulo R a ring. Then,

$$H^i(\mathbb{C}P^k, R) = \begin{cases} R & i = 2n, n \leqslant k \\ 0 & \textit{otherwise} \end{cases}$$

Proof of Corollary 4.2.4:
This comes from the fact that $\pi_i(HR) = R$ if $i = 0$ and $\pi_i(HR) = 0$ otherwise. $\square$ 4.2.4

Corollary 4.2.5.
Let E be a complex oriented spectrum.

$$E^i(MU(1)) \cong E^i(\mathbb{C}P^\infty, R) \cong \bigoplus_{j=1}^{\infty} \pi_{i-2j}(E) \textit{ for } E^* \textit{ reduced}$$

Proof of Corollary 4.2.5:

$$MU(1) = \operatorname{colim}_k T(\gamma_{1,k}) = \operatorname{colim}_k \mathbb{C}P^{k+1} = \mathbb{C}P^\infty$$

Then, we see that $\{\mathbb{C}P^n\}_{n\in\mathbb{N}}$ follows the Mittag-Leffler criterion 1.4.22. Thus

$$H^i(\mathbb{C}P^\infty, R) \cong \operatorname{colim}_k H^i(\mathbb{C}P^k, R) \cong \operatorname{colim}_k \bigoplus_{j=1}^{k} \pi_{i-2j}(E) = \bigoplus_{j=1}^{\infty} \pi_{i-2j}(E).$$

$\square$ 4.2.5

This also gives us a structure on the cohomology ring.

Theorem 4.2.6.
Let E be an oriented cohomology

$$E^\bullet(\mathbb{C}P^k) \cong \pi_\bullet(E)[u]/(u+1)$$

$$E^\bullet(\mathbb{C}P^k) \cong \pi_\bullet(E)[u]$$

with $u \in E^2(\mathbb{C}P^k)$ being the Thom class given by the orientability of E.

If we have already the abelian group structure, proving the ring structure is quite complex, using notion of Atiyah-Hirzebruch spectral sequence. The proof can be found in [Ped18], 2.0.4.

4.2.2 Hurewitz fibrations and Chern class.

Now that we have an idea of how complex it is to compute in the general case, we will restrict ourselves to computing $H\mathbb{Z}^*(MU)$. To do so, we need to define what are called fibrations.

Definition 4.2.7 (Hurewitz fibrations)**.**
Let $E, B \in Ob(\mathbf{CW})$, $p \in \mathbf{CW}(E, B)$. It is called ***a Hurewitz fibration*** *if $\forall \tilde{f} : Z \to E$, such that $p \circ \tilde{f} \sim_{Hom} g$ using H, $\exists \tilde{H} : Z \times [0,1]^+ \to E$ such that $\tilde{H}_0 = \tilde{f}$, $p \circ \tilde{H} = H$. I.e. the following diagram commute:*

$$\begin{array}{ccc} Z & \xrightarrow{\quad \tilde{f} \quad} & E \\ {\scriptstyle id \times 0}\downarrow & \nearrow^{\exists \tilde{H}} & \downarrow {\scriptstyle p} \\ Z \wedge [0,1]^+ & \xrightarrow{\quad H \quad} & X \end{array}$$

We call $F = p^{-1}(x_0)$ the ***fibre*** *of our Hurewitz fibration.*

Example 4.2.8.
Many examples of fibrations can be found in [Swi17], chapter 4.

- *Any product $B \times F \xrightarrow{p_1} B$, using $\tilde{H} = (H(x,t), p_2 \circ f(x))$.*
- *Let $E \xrightarrow{p} B$. For $f : Z \to E$, consider $H : Z \wedge [0,1]^+ \to B$ an homotopy of $p \circ f$.*

  Using $\{U_k\}$ trivial covering on B, we get an open covering $\{V_k \times I_k\}$ on $Z \wedge [0,1]^+$. On any open, we can define by triviality a lift $\tilde{H}_k$. Because $[0,1]^+$ is compact, for each $y \in Z$, $\exists V_y \times [0,1]$ such that we have a lift $\tilde{H}_y$ on it.

  To see this point, we have $y \times [0,1]$ is cover by finitely many open $V_k \times I_k$. By setting $V_y = \bigcap_k V_k$, we get $V_y \times [0,1]$. Furthermore, we can merges our lifts $\tilde{H}_k$ together to get $\tilde{H}_y$.

  Now, we have an open cover with lift well defined that. Thus, we can merge everything together and get $\tilde{H}$.

  $$F \to E \to B \text{ is thus a fibration}$$

- *Any quotient group given by a topological group.*

  $$H \to G \xrightarrow{\pi} G/H$$

- *Path space fibration using $PH = Hom\big(([0,1], 0), X\big)$ and using $\pi(f) = f(1)$*

  $$\Omega X \to PX \xrightarrow{\pi} X$$

Theorem 4.2.9.
Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ *with* $F = p^{-1}(x_0)$ *the fibre of our Hurewitz fibration. We have the following long exact sequence*

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

Proof of Theorem 4.2.9:
This comes from the fact that, similarly to cofibre, we have a long sequence where every triple is a Hurewitz fibration ([Swi17] 4.42, [Hat02] page 409),

$$\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

Now, consider $F \to E \to B$ a Hurewitz fibration. Let's show that

$$\pi_k(F) \xrightarrow{\iota_*} \pi_k(E) \xrightarrow{p_*} \pi_k(B)$$

is exact:

- $Im(\iota_*) \subseteq \ker(p_*)$: By definition, $(p \circ \iota \circ f) = x_0$.
- $Im(\iota_*) \supseteq \ker(p_*)$: Let $f : S^k \to E$ such that $[p \circ f] = 0$ using H. Then, by fibration, we get $\tilde{H} : S^k \wedge [0,1]^+$, $p \circ \tilde{H} = H$. Let $\tilde{H}_1 = \tilde{f}$. We get that $f \sim_{Hom} \tilde{f}$, $p \circ \tilde{f} = *$. Therefore, $\tilde{f}(S^k) \subseteq F$. We have
$$\tilde{f} : S^k \to F$$
with $\iota \circ \tilde{f} \sim_{Hom} f$, proving our point.

□ 4.2.9

Now, let's assume the following technical theorem.

Theorem 4.2.10 (Thom-Gysin sequence)**.**
Let $S^n \to E \to B$ *be an Hurewitz fibration over a simply connected CW complex* [8], R *be a commutative ring. Then,* $\exists c \in H^{n+1}(E, R)$ *such that we get the following long exact sequence:*

$$\cdots H^k(B,R) \xrightarrow{p^*} H^k(E,R) \longrightarrow H^{k-n}(B,R) \xrightarrow{c \smile (-)} H^{k+1}(B,R) \to \cdots$$

Proof given in [Swi17], 15.30.

[8] This means $\pi_i(B) = 0$ for $i \leqslant n$.

Now that we have those definitions, let's compute $H^i(BU(n))$. To do so, let's first define the unitary group.

Definition 4.2.11 (Unitary group)**.**
We define the ***unitary group*** $U(n) \subset GL_n(\mathbb{C})$*:*

$$U(n) = \Big\{ M \in GL_n(\mathbb{C}) |\ MM^* = id_n \Big\}$$

with M^* *being the Hermitian transpose.*

Proposition 4.2.12.

$$G^{\mathbb{C}}_{n,k} \cong U(n+k) \Big/ \big(U(n) \times U(k) \big)$$

We thus get two Hurewitz fibrations,

$$U(n) \to U(n+k)/U(k) \to G^{\mathbb{C}}_{n,k}$$

$$U(k) \to U(n+k) \to U(n+k)/U(k) = V_n(\mathbb{C}^{n+k})$$

We name $V_n(\mathbb{C}^{n+k})$ *the* ***Stiefel manifold.***

Proof of Proposition 4.2.12:
To see this isomorphism, we have that any k vector space in $\mathbb{C}^{n+k}$ is given by a orthonormal basis, i.e. an element of $U(n+k)$. But the vector space is unchanged by any modification of its basis or its complementary space, i.e. a linear action on $U(n) \times U(k)$. Thus,

$$G^{\mathbb{C}}_{n,k} \cong U(n+k) \Big/ \big(U(n) \times U(k) \big)$$

The fact that we have Hurewitz fibration is a consequence of 4.2.8.

□ 4.2.12

Now, using our fibrations, let's find some useful results on unitary group.

Proposition 4.2.13.
Let $N \in \mathbb{N}$*.* $\forall n \leqslant N$*,* $i : U(n) \hookrightarrow U(N)$ *induce an isomorphism*

$$i_* : \pi_j\big(U(n)\big) \cong \pi_j\big(U(N)\big)$$

for $j < 2n$*.*
Furthermore, $i_* : \pi_{2n}\big(U(n)\big) \twoheadrightarrow \pi_{2n}\big(U(N)\big)$*.*

Proof of Proposition 4.2.13:
wlog, we consider $N = n+1$. Then, we have the following Hurewitz fibration

$$U(n) \to U(n+1) \to U(n+1)/U(n) = S^{n+1}_{\mathbb{C}} = S^{2n+1}_{\mathbb{R}}$$

Thus, using 4.2.9, we get that the l.e.s.

$$\cdots \to \pi_{j+1}(S^{2n+1}) \to \pi_j(U(n)) \xrightarrow{i^*} \pi_j(U(n+1)) \to \pi_j(S^{2n+j}) \to \cdots$$

For $j < 2n$, $\pi_j\big(U(n)\big) \cong \pi_j\big(U(n+1)\big)$ and $\pi_{2n}\big(U(n)\big) \twoheadrightarrow \pi_{2n}\big(U(n+1)\big)$

□ 4.2.13

Those results have consequences on the Stiefel manifold.

Proposition 4.2.14.

$$\pi_i\big(V_n(\mathbb{C}^{n+k})\big) = 0$$

for $i < 2k$.

Proof of Proposition 4.2.14:
Using the following Hurewitz fibration

$$U(k) \xrightarrow{i} U(n+k) \to V_n(\mathbb{C}^{n+k})$$

we get using 4.2.9 the following long exact sequence

$$\cdots \to \pi_{j+1}(U(k)) \xrightarrow{i_*} \pi_{j+1}(U(n+k)) \to \pi_{j+1}(V_n(\mathbb{C}^{n+k}) \to \pi_j(U(k)) \xhookrightarrow{i_*} \pi_j(U(k+n)) \to \cdots$$

Thus, for $j < 2k$, using 4.2.13, we get that

$$\pi_j(V_n(\mathbb{C}^{n+k})) = 0.$$

□ 4.2.14

Corollary 4.2.15.

$$\pi_i(G^{\mathbb{C}}_{n,k}) \cong \pi_{i-1}(U(n))$$

for $i < 2k$.

Proof of Corollary 4.2.15:
Using the Hurewitz fibration $U(n) \to V_n(\mathbb{C}^{n+k}) \to G^{\mathbb{C}}_{n,k}$ and 4.2.9, we get the long exact sequence

$$\cdots \to \pi_j(V_n(\mathbb{C}^{n+k})) \to \pi_j(G^{\mathbb{C}}_{n+k}) \to \pi_{j-1}(U(n)) \to \pi_{j-1}(V_n(\mathbb{C}^{n+k})) \to \cdots$$

Because $\pi_i(V_n(\mathbb{C}^{n+k})) = 0$, $i < 2k$ we get $\pi_i(G^{\mathbb{C}}_{n,k}) \cong \pi_{i-1}(U(n))$. □ 4.2.15

Corollary 4.2.16.
Let $EU(n) = \ colim_k V_n(\mathbb{C}^{n+k})$.

$EU(n)$ is contractible.

Proof of Corollary 4.2.16:
This comes from the fact that $\pi_i(\ \mathrm{colim}_k X_k) \cong\ \mathrm{colim}_k \pi_i(X_k)$ by 1.4.19. Thus $\forall i \in \mathbb{N}$, because $\pi_i(V_n(\mathbb{C}^{n+k})) = 0$ for $i < 2k$, we get tat $\pi_i(EU(n)) = 0$. Thus,

$$\iota_* : \pi_i(*) \cong \pi_i(EU(n))$$

Thus, ι is a weak homotopy equivalence and using Whitehead theorem 1.4.17, it is in fact an homotopy equivalence.

□ 4.2.16

Now, let's show an interesting result that put greater light on why BU is called BU.

Proposition 4.2.17.
Let $F \to E \to B$ be an Hurewitz fibration with E contractible set. Then, there exists a weak homotopy equivalence $F \to \Omega B$.

Proof of Proposition 4.2.17:
To do so, we consider PB, the set of all paths ν in B from any x to x_0 the base point. By definition, PB is contractible.
Because E is contractible, consider H_t the homotopy between id_E and $*$. We use it to define a morphism

$$q : E \to PB$$
$$q(x) = p(H_t(x))$$

By restriction, because $F = p^{-1}(x_0)$, we get that $q|_F : F \to \Omega B$. We have that by 4.2.8 that $\Omega B \to PB \to B$ is a Hurewitz fibration. Thus, we have the following diagram:

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle q|_F} & & \downarrow{\scriptstyle q} & & \| \\ \Omega B & \longrightarrow & PB & \longrightarrow & B \end{array}$$

Using 4.2.9 twice, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
\pi_{j+1}(E) & \longrightarrow & \pi_{j+1}(B) & \longrightarrow & \pi_j(F) & \longrightarrow & \pi_j(E) & \longrightarrow & \pi_j(B) \\
\Big\Downarrow q_* & & \Big\Downarrow id_* & & \Big\downarrow (q|_F)_* & & \Big\Downarrow q_* & & \Big\Downarrow id_* \\
\pi_{j+1}(PE) & \longrightarrow & \pi_{j+1}(B) & \longrightarrow & \pi_j(\Omega B) & \longrightarrow & \pi_j(PE) & \longrightarrow & \pi_j(B)
\end{array}$$

The equality $\pi_j(E) \cong \pi_j(PE)$ being because they are both 0. Using Five lemma, we get that $q|_F$ is a weak homotopy equivalence. □ 4.2.17

Corollary 4.2.18.

$$\Omega BU(n) \sim_{Hom} U(n).$$

Proof of Corollary 4.2.18:
Going to colimit, we get the following Hurewitz fibration

$$U(n) \to EU(n) \to BU(n)$$

Thus, using 4.2.17, we get a weak homotopy equivalence that is just an homotopy equivalence using Whitehead thorem 1.4.17. □ 4.2.18

Now, let's sow this important technical lemma.

Lemma 4.2.19.
There exists the following Hurewitz fibration on $BU(n)$:

$$S^{2n+1}_{\mathbb{R}} \to BU(n) \to BU(n+1).$$

Proof of Lemma 4.2.19:
To prove this lemma, Let $\widetilde{BU(n)} = EU(n+1)/U(n)$. We have the following diagram:

$$\begin{array}{ccccc}
U(n) & \longrightarrow & EU(n) & \longrightarrow & BU(n) \\
\Big\| & & \Big\downarrow & & \Big\downarrow \\
U(n) & \longrightarrow & EU(n+1) & \longrightarrow & \widetilde{BU(n)}
\end{array}$$

The two rows are Hurewitz fibrations. Using 4.2.9, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
\pi_{j+1}(U(n)) & \longrightarrow & \pi_{j+1}(EU(n)) & \longrightarrow & \pi_j(BU(n)) & \longrightarrow & \pi_j(U(n)) & \longrightarrow & \pi_j(EU(n)) \\
\Big\Downarrow id_* & & \Big\Downarrow & & \Big\downarrow & & \Big\Downarrow id_* & & \Big\Downarrow \\
\pi_{j+1}(U(n)) & \longrightarrow & \pi_{j+1}(EU(n+1)) & \longrightarrow & \pi_j(\widetilde{BU(n)}) & \longrightarrow & \pi_j(U(n)) & \longrightarrow & \pi_j(EU(n+1))
\end{array}$$

Thus, using Five lemma and 1.4.17, we get that $BU(n) \sim_{Hom} \widetilde{BU(n)}$.
Now, we simply have the following Hurewitz fibration

$$U(n+1)/U(n) \to EU(n+1)/U(n) \to EU(n+1)/U(n+1)$$

$$S_{\mathbb{R}}^{2n+1} \to \widetilde{BU(n)} \to BU(n+1)$$

□ 4.2.19

Having this lemma and Thom-Gysin sequence theorem 4.2.10, we now can compute the structure of $H^*(BU(n))$ using Chern class.

Theorem 4.2.20 (Chern class theorem)**.**
Let H be the standard unreduced cohomology.

$$H^\bullet(BU(n)) \cong \mathbb{Z}[c_1^{(n)}, \cdots, c_n^{(n)}]$$

*with $c_i \in H^{2i}(BU(n))$ beeing **Chern class** such that:*

1. $c_0^{(n)} = 1$
2. $c_1^{(1)}$ *is given by the orientation of $H^2(CP^\infty)$, using the fact that $MU(1) = BU(1)$.*
3. *using $i : BU(n) \to BU(n+1)$, we get $i^* c_j^{(n+1)} = c_j^{(n)}$.*
4. *using $m : BU(n) \times BU(m) \to BU(n+m)$, we get $c_k^{(n+m)} = \sum_{i+j=k} c_i^{(n)} \cup c_j^{(m)}$.*

Proof of Theorem 4.2.20:
We will prove this theorem by induction: $n = 1$ given by 4.2.6.

Suppose $H^\bullet(BU(n)) \cong \mathbb{Z}[c_1^{(n)}, \cdots, c_n^{(n)}]$. Then, let's consider the $n+1$ case. We have by 4.2.19 the following Hurewitz fibration:

$$S_{\mathbb{R}}^{2n+1} \to BU(n) \to BU(n+1)$$

Using Thom-Gysin sequence 4.2.10, we get that the following long exact

$$\cdots \to H^k(BU(n+1)) \to H^k(BU(n)) \to H^{k-2n-1}(BU(n+1)) \to H^{k+1}(BU(n+1)) \to \cdots$$

Let's use it to compute some cohomology:

- By nature of $H^*(-)$ and because $BU(n)$ is connected, we get that $\forall n \in \mathbb{N}^+$,

  $$H^{-n}(BU(k)) = 0$$

  $$H^0(BU(k)) = \mathbb{Z}$$

  To see that $BU(n)$ is connected, we use 4.2.13 and $0 = \pi_0(S^1) = \pi_0(U(1)) \cong \pi_0(U(n))$. Then, by using 4.2.9, we have

  $$\pi_0(U(n)) \to \pi_0(EU(n)) \to \pi_0(BU(n)) \to 0$$

  Thus, $\pi_0(BU(n)) \cong \pi_0(EU(n)) = 0$.

  Note that, using 4.2.18, we get that

  $$\pi_1(BU(n)) \cong \pi_0(U(n)) = 0$$

- $2k+1 < 2(n+1)$:

  $$\cdots \to H^{2k}(BU(n)) \to H^{2k-2n-1}(BU(n+1)) \to H^{2k+1}(BU(n+1)) \to H^{2k+1}(BU(n)) \to \cdots$$

  Using the fact that, by induction, $H^{2k+1-2n-1}(BU(n+1)) = H^{2k+1}(BU(n)) = 0$, we get that

  $$H^{2k+1}(BU(n+1)) = 0$$

- $2k+1 > 2(n+1)$: Consider

  $$\cdots \to H^{2k-2n-1}(BU(n+1)) \to H^{2k+1}(BU(n+1)) \to H^{2k+1}(BU(n)) \to \cdots$$

  By induction, $H^{2k+1}(BU(n)) = 0$. Thus,

  $$H^{2k-2n-1}(BU(n+1)) \twoheadrightarrow H^{2k+1}(BU(n+1)).$$

  But then, using another induction, we can guaranty that $H^{2k-2n-1}(BU(n+1)) = 0$. Thus $H^{2k+1}(BU(n+1)) = 0$

- $2k$: Consider the following exact sequence

  $$\cdots \to H^{2k-1}(BU(n)) \to H^{2k-2(n+1)}(BU(n+1)) \to H^{2k}(BU(n+1)) \to H^{2k}(BU(n)) \to H^{2k-2n-1}(BU(n+1)) \to \cdots$$

  Using $H^{2k-1}(BU(n)) = H^{2k-2n-1}(BU(n+1)) = 0$, we get for all k the s.e.s.

  $$0 \to H^{2k-2(n+1)}(BU(n+1)) \to H^{2k}(BU(n+1)) \to H^{2k}(BU(n)) \to 0$$

– If $k < n+1$, get

$$0 \to 0 \to H^{2k}(BU(n+1)) \xrightarrow{i^*} (\mathbb{Z}[c_1, \cdots, c_n])_{2k} \to 0$$

Thus

$$H^{2k}(BU(n+1)) \cong \mathbb{Z}[c_1, \cdots, c_n])_{2k} \cong (\mathbb{Z}[c_1, \cdots, c_{n+1}])_{2k}$$

– If $k = n+1$: We are in the following representation.

$$0 \to \mathbb{Z} \xrightarrow{c_{n+1} \smile -} H^{2(n+1)}(BU(n+1)) \xrightarrow{i^*} (\mathbb{Z}[c_1, \cdots, c_n])_{2(n+1)} \to 0$$

Using the fact that $\mathbb{Z}$ and $(\mathbb{Z}[c_1, \cdots, c_n])_{2k}$ can be seen as free $\mathbb{Z}$ modules, we get that $H^{2k}(BU(n+1))$ split and is therefore free. Thus,

$$H^{2(n+1)}(BU(n+1)) \cong (\mathbb{Z}[c_1, \cdots, c_n])_{2(n+1)} \oplus \mathbb{Z}(c_{n+1})$$

$$H^{2(n+1)}(BU(n+1)) \cong (\mathbb{Z}[c_1, \cdots, c_{n+1}])_{2(n+1)}$$

– If $k > n+1$. We get that

$$0 \to (\mathbb{Z}[c_1, \cdots, c_{n+1}])_{2k-2(n+1)} \xrightarrow{c_{n+1} \smile -} H^{2k}(BU(n+1)) \xrightarrow{i^*} (Z[c_1, \cdots, c_n])_{2k} \to 0$$

It also split in

$$H^{2n}(BU(n+1)) \cong (\mathbb{Z}[c_1, \cdots, c_n])_{2k} \oplus \Big(c_{n+1} \smile (\mathbb{Z}[c_1, \cdots, c_{n+1}])_{2k-2(n+1)}\Big)$$

$$H^{2n}(BU(n+1)) \cong (\mathbb{Z}[c_1, \cdots, c_{n+1}])_{2n}$$

Thus, we get our desired result

$$H^\bullet(BU(n)) \cong \mathbb{Z}[c_1, \cdots, c_n]$$

□ 4.2.20

Example 4.2.21 (Tabular of unreduced cohomology of BU).

	H^0	H^2	H^4	H^6	H^8	H^{10}	H^{12}	H^{14}	H^{16}	H^{18}	H^{20}	H^{22}	H^{24}	H^{26}	H^{28}
$BU(0)$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$BU(1)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$BU(2)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^5$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^7$	$\mathbb{Z}^7$	$\mathbb{Z}^8$
$BU(3)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{16}$	$\mathbb{Z}^{19}$	$\mathbb{Z}^{21}$	$\mathbb{Z}^{24}$
$BU(4)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^6$	$\mathbb{Z}^9$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{15}$	$\mathbb{Z}^{18}$	$\mathbb{Z}^{23}$	$\mathbb{Z}^{27}$	$\mathbb{Z}^{34}$	$\mathbb{Z}^{39}$	$\mathbb{Z}^{47}$
$BU(5)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{13}$	$\mathbb{Z}^{18}$	$\mathbb{Z}^{23}$	$\mathbb{Z}^{30}$	$\mathbb{Z}^{37}$	$\mathbb{Z}^{47}$	$\mathbb{Z}^{57}$	$\mathbb{Z}^{70}$
$BU(6)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{20}$	$\mathbb{Z}^{26}$	$\mathbb{Z}^{35}$	$\mathbb{Z}^{44}$	$\mathbb{Z}^{58}$	$\mathbb{Z}^{71}$	$\mathbb{Z}^{90}$

This tabular is given by the following recursive formula $H^{2k}(BU(n)) = \mathbb{Z}^{\alpha_n(k)}$ with

$$\alpha_n(k) = \begin{cases} \alpha_n(k-n) + \alpha_{n-1}(k) \\ \delta_{0,k}, n = 0 \\ 0, k < 0 \end{cases}$$

Corollary 4.2.22.
Let $H\mathbb{Z}$ be the standard reduced cohomology. Then

$$H\mathbb{Z}^k(MU(n)) = H^{k-2n}(BU(n))$$

Proof of Corollary 4.2.26:
Using Thom-Dolt isomorphism 4.1.15.

□ 4.2.26

Example 4.2.23 (Tabular of reduced cohomology of MU)**.**

	$H\mathbb{Z}^0$	$H\mathbb{Z}^2$	$H\mathbb{Z}^4$	$H\mathbb{Z}^6$	$H\mathbb{Z}^8$	$H\mathbb{Z}^{10}$	$H\mathbb{Z}^{12}$	$H\mathbb{Z}^{14}$	$H\mathbb{Z}^{16}$	$H\mathbb{Z}^{18}$	$H\mathbb{Z}^{20}$
$MU(0)$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
$MU(1)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$MU(2)$	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^5$
$MU(3)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$
$MU(4)$	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^6$	$\mathbb{Z}^9$	$\mathbb{Z}^{11}$
$MU(5)$	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^{10}$
$MU(6)$	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^5$	$\mathbb{Z}^7$

We can in fact generalised Chern class on any complex oriented ring spectrum.

Theorem 4.2.24 (Conner-Floyd Chern class)**.**
Let E be a complex oriented ring spectrum. We get that

$$E^\bullet(BU(n)) \cong \pi_\bullet(E)[c_1^E, \cdots, c_n^E]$$

with $c_i^E \in E^{2i}(BU(n))$ the ***Conner-Floyd Chern class****, having similar properties than the Chern class.*

The proof is using Atiyah-Hirzebuch spectral sequence and is thus too complex for the scope of this projet, but a proof can be found in [CF06].

Corollary 4.2.25.
Let E be a complex oriented ring spectrum.

$$E^k(MU(n)) \cong E^{k-2n}(BU(n)) \oplus E^{k-2n}(S^0)$$

Proof of Corollary 4.2.25:
By Thom-Dolt isomorphism 4.1.15.

□ 4.2.25

Let's now compute the cohomology of the MU spectrum using the following definition 2.1.47.

Proposition 4.2.26.
Let E be a complex oriented ring spectrum. We get

$$E^k(MU) \cong \ lim_n \ E^k(BU(n)^+).$$

Proof of Proposition 4.2.26:
Using the fact that $MU_{-k} = *$, we get

$$\begin{array}{rcl} E^k(MU) & = & \lim_k \ E^{k+2n}(MU(n)) \\ & \cong & \lim_k \ E^k(BU(n)^+)^9 \end{array}$$

□ 4.2.26

Proposition 4.2.27.

$$H^{2k+1}(MU) = 0$$
$$H^{2k}(MU) \cong H^{2k}(BU(k))$$

Proof of Proposition 4.2.27:
This comes from the fact that $H^{2n+1}(BU(n)) = 0$.
Otherwise, we have by induction on t that $\alpha_{n+t}(n) = \alpha_n(n)$, with $\alpha_n(k)$ given in 4.2.21. Thus,

$$\lim_n H^{2k}(BU(n)) = \ \lim_n \ \mathbb{Z}^{\alpha_n(k)} = \mathbb{Z}^{\alpha_k(k)} = H^{2k}(BU(k)).$$

□ 4.2.27

[9] Using 4.2.25

Appendix A

To go further in the study of MU

In this section, we will state other interesting results about MU that we don't have time to study in details in this paper.

A.1 Computing $\pi_*(MU)$

First, let's have a look at $\pi_*(MU)$. If we won't prove its full structure, we will give a first intuition, using complex cobordism that we proved in 3.2.26.
To do so, consider the following lemma

Lemma A.1.1.
Let $X \in Ob(\mathbf{CW})$ be compact, A be an abelian group. Then

$$H^k(X, A) = 0 \text{ except on finitely many cases.}$$

Furthermore, $\sum_{k \in \mathbb{N}} \; dim \; H^k(X, \mathbb{F}_2) < \infty$.

*Proof of Lemma A.*1.1:
Because X is compact, it is a finite cell complex. Using the cofibre sequence on the skeleton

$$X^k \to X^{k+1} \to \vee_{i=1}^{n_k} S_i^{k+1}$$

we get the l.e.s

$$\cdots \leftarrow H^{i+1}(\vee_{j=1}^{n_k} S_j^{k+1}) \leftarrow H^i(X^k, A) \leftarrow H^i(X^{k+1}, A) \leftarrow H^i(\vee_{j=1}^{n_k} S_j^{k+1}, A) \leftarrow \cdots$$

Let us show by recursively that $\forall i > k, H^i(X^k, R) = 0$. Because $X^0 = \vee_j S_j^0$, the initiation is true. Then, if we assume $\forall i > k, H^i(X^k, R) = 0$, we get that

$$0 \leftarrow H^i(X^k, R) \leftarrow H^i(X^{k+1}, R) \leftarrow 0$$

for $i > k+1$. Therefore, using our recurrence hypothesis, we get that $H^i(X^{k+1}, R) = 0$, $i > k$. Thus, because X compact, $X = X^k$. Thus, $H^i(X, R) = 0$ for $i > k$.

Now, to show that $\sum_{k\in\mathbb{N}} \dim H^k(X, \mathbb{F}_2) < \infty$, by induction.
$H^k(X^0, R) = H^k(\vee_j S^0_j, \mathbb{F}_2) = \mathbb{F}_2^j$ if $k = 0$, 0 otherwise. Thus, $\sum_{k\in\mathbb{N}} \dim H^k(X^0, \mathbb{F}_2) = j < \infty$
By the long exact sequence, we have

$$0 \leftarrow H^{k+1}(X^{k+1}, \mathbb{F}_2) \leftarrow \mathbb{F}_2^{n_k} \leftarrow H^k(X^k, \mathbb{F}_2) \leftarrow H^k(X^{k+1}, \mathbb{F}_2) \leftarrow 0$$

Thus, $\dim H^{k+1}(X^{k+1}, \mathbb{F}_2) < n_k$ and $\dim H^k(X^{k+1}, \mathbb{F}_2) < \dim H^k(X^k, \mathbb{F}_2)$.
Furthermore, $H^k(X^{k+1}, \mathbb{F}_2) \cong H^k(X^k, \mathbb{F}_2), i \neq k, k+1$. Thus

$$\sum_{k\in\mathbb{N}} \dim H^k(X^{k+1}, \mathbb{F}_2) = j < \infty$$

□ *A.1.1*

We now can define what is called the Euler characteristic.

Definition A.1.2 (Euler characteristic mod 2)**.**
Let $X \in Ob(\mathbf{CW})$ be compact. Using previous lemma A.1.1, we define the ***Euler characteristic (mod 2)*** *as:*

$$\chi_2(X) = \sum_{k\in\mathbb{N}} (-1)^k \dim\left(H^k(X^+, \mathbb{F}_2)\right).$$

Example A.1.3.

$$\chi_2(S^{2n+1}_{\mathbb{R}}) = 0$$
$$\chi_2(S^{2n}_{\mathbb{R}}) = 2$$
$$\chi_2(\mathbb{C}P^k) = k+1$$

Euler characteristic has some good properties

Proposition A.1.4.
Let $X, Y \in Ob(\mathbf{CW})$ be compacts. Then,

$$\chi_2(X \vee Y) = \chi_2(X) + \chi_2(Y)$$

$$\chi_2(X \wedge Y) = \chi_2(X).\chi_2(Y)$$

Proof of Proposition A.1.4:

$$\begin{array}{rcl}\chi_2(X \vee Y) & = & \sum_{k\in\mathbb{N}}(-1)^i \dim\left(H^k(X \sqcup Y, \mathbb{F}_2)\right) \\ & = & \sum_{k\in\mathbb{N}}(-1)^i \dim\left(\left(H^k(X,\mathbb{F}_2)\right) \times \left(H^k(Y,\mathbb{F}_2)\right)\right) \\ & = & \chi_2(X) + \chi_2(Y)\end{array}$$

$$\begin{array}{rcl}\chi_2(X \wedge Y) & = & \sum_{k\in\mathbb{N}}(-1)^k \dim\left(H^k(X \times Y, \mathbb{F}_2)\right) \\ & = & \sum_{k\in\mathbb{N}}(-1)^k \dim\left(\sum_{i+j=k} H^i(X,\mathbb{F}_2) \otimes H^j(Y,\mathbb{F}_2)\right)^1 \\ & = & \sum_{k\in\mathbb{N}}\sum_{i+j=k}(-1)^i(-1)^j \dim\left(H^i(X,\mathbb{F}_2)\right).\dim\left(H^j(Y,\mathbb{F}_2)\right) \\ & = & \chi_2(X).\chi_2(Y)\end{array}$$

□ *A.1.4*

Lemma A.1.5.
Let $X, Y, Z \in Ob(\mathbf{CW})$ be compact spaces such that $X \to Y \to Z$ is a cofibre sequence. Then, the Euler characteristic is defined and

$$\chi_2(Y) = \chi_2(X) + \chi_2(Z)$$

Proof of Lemma A.1.5:
Consider the long exact sequence for cohomology induced by the cofibre sequence.

$$\cdots \leftarrow H^{k+1}(X^+,\mathbb{F}_2) \xleftarrow{\partial_k} H^k(Z^+,\mathbb{F}_2) \xleftarrow{a_k} H^k(Y^+,\mathbb{F}_2) \xleftarrow{b_k} H^k(X^+,\mathbb{F}_2) \xleftarrow{\partial_{k-1}} H^{k-1}(Z^+,\mathbb{F}_2) \leftarrow \cdots$$

Then, because abelian group are $\mathbb{F}_2$-linear group, we get, using exactness that:

$$d_Z^k = \dim H^k(Z^+,\mathbb{F}_2) = \dim(\partial_k) + \dim(a_k)$$

$$d_Y^k = \dim H^k(Y^+,\mathbb{F}_2) = \dim(a_k) + \dim(b_k)$$

$$d_X^k = \dim H^k(X^+,\mathbb{F}_2) = \dim(a_k) + \dim(\partial_{k-1})$$

$$\text{Thus, } d_X^k + d_Z^k - d_Y^k = \dim(\partial_k) + \dim(\partial_{k-1})$$

Therefore, we get

$$\begin{array}{rcl}\chi_2(X) + \chi_2(Z) - \chi_2(Y) & = & \sum_{k\in\mathbb{N}}(-1)^k\left(d_X^k + d_Z^k - d_Y^k\right) \\ & = & \sum_{k\in\mathbb{N}}(-1)^k(\dim(\partial_k) + \dim(\partial_{k-1})) \\ & = & 0\end{array}$$

We get to zero because $\dim(\partial_0) = 0$ and ∂_k become zero at some point. □ *A.1.5*

[1] This is given by Kunneth theorem which can be found in [Swi17], 13.13.

Corollary A.1.6.
Let X, Y be compact CW complexes.

$$\chi_2(X \cup Y) = \chi_2(X) + \chi_2(Y) - \chi_2(X \cap Y)$$

Proof of Corollary A.1.6:
This is because we have the following cofibre:

$$(X \cap Y)^+ \to X^+ \vee Y^+ \to (X \cup Y)^+$$

□ *A*.1.6

Now, let us assume the following theorem

Theorem A.1.7 (Poincare duality)**.**
Let M be a compact n manifold without boundary. $\forall k$

$$H_k(M, \mathbb{F}_2) \cong H^{n-k}(M, \mathbb{F}_2)$$

Furthermore, using universal coefficient theorem, we get that

$$\dim H^k(M, \mathbb{F}_2) = \dim H^{n-k}(M, \mathbb{F}_2)$$

Proof given in [Swi17], 14.13.

This induce the following idea.

Corollary A.1.8.
Let M be a compact odd dimensional manifold without boundary. Then

$$\chi_2(M) = 0$$

Proof of Corollary A.1.8:
Because M is a compact manifold, we have that $H^k(M, \mathbb{F}_2) \neq 0 \Rightarrow k = 0, \cdots, n$. Now, we have that

$$\begin{aligned} 2\chi_2(M) &= \sum_{k=0}^{n} (-1)^k \dim \left(H^k(M, \mathbb{F}_2)\right) + \sum_{k=0}^{n} (-1)^k \dim \left(H^{n-k}(M, \mathbb{F}_2)\right) \\ &= \sum_{k=0}^{n} (-1)^k \Big(\dim \left(H^k(M, \mathbb{F}_2)\right) - \dim \left(H^k(M, \mathbb{F}_2)\right)\Big) \\ &= 0 \end{aligned}$$

□ *A.1.8*

Using this corollary, we can get the following result on complex projective spaces.

Lemma A.1.9.

$$\mathbb{C}P^{2k} \text{ is no boundary of a compact manifold}$$

Proof of Lemma A.1.9:

- Suppose $\mathbb{C}P^{2k} = \partial W$ with W a compact $2k+1$ manifold with boundary. Then, consider $W \#_{\partial W} W$, a compact $2k+1$ manifold without boundary. We get that

$$\begin{aligned} 0 &= \chi_2(W \#_{\partial W} W) \\ &= 2\chi_2(M) - \chi_2(\mathbb{C}P^{2k}) \\ &= 2\chi_2(M) - 2k + 1 \end{aligned}$$

  Thus, 0 is the sum of an odd and even number, which is a contradiction.

□ *A.1.9*

Corollary A.1.10.

$$\forall k \in \mathbb{N}, \pi_{2k}(MU) \neq 0$$

Proof of Corollary A.1.10:
Using 3.2.26, we have that $\pi_{2n}(MU) \cong \Omega_{2n}(*)$.
Now, we have $[\mathbb{C}P^{2k}] \in \Omega_{2n}(*)$ and $[\mathbb{C}P^{2k}] \neq 0$ using A.1.9. Thus, $\Omega_{2n}(*) \neq 0$.

□ *A.1.10*

In fact, $\pi_n(MU)$ has the following structure

Theorem A.1.11 (Milnor-Novikov theorem)**.**

$$\pi_\bullet(MU) \cong \mathbb{Z}[b_1, \cdots b_n, \cdots]$$

with $b_i \in \pi_{2i}(MU)$, $b_i = \Psi(\mathbb{C}P^i)$.

This theorem is proven in [Rav03], Theorem 3.1.5.

A.2 Nilpotence theorem

Another very important property of the spectrum MU is the nilpotence theorem. It is in fact central in all further study of spectral homotopy theory.

Definition A.2.1 (Hurewicz map).
Let $R \in Ob(\mathbf{Sp})$ and let E be a ring spectrum. Then, we define the ***Hurewicz map****:*

$$h : \pi_k(R) \to E_k(R)$$

$$\pi_k(R) \cong \pi_k(R \wedge \mathbb{S}) \xrightarrow{id_R \wedge \eta} \pi_k(R \wedge E) = E_k(R)$$

In fact, this map extend to the homotopy ring.

$$h : \pi_\bullet(R) \to E_\bullet(R)$$

Theorem A.2.2 (Nilpotence theorem).
Let R be a ring spectrum. Consider the Hurewicz map

$$\pi_\bullet(R) \xrightarrow{h} MU_\bullet(R)$$

Then, $\alpha \in \pi_\bullet(R)$ is nilpotent to multiplication $\iff h(\alpha) = 0$

Proof in [Rav92], Chapter 9.

Corollary A.2.3 (Nishida).

$$\forall \alpha \in \pi_n(\mathbb{S}), n \neq 0, \alpha \text{ is nilpotent.}$$

Proof of Corollary A.2.3:
Let $x \in \pi_n(\mathbb{S})$. If $n < 0$, then $\pi_n(\mathbb{S}) = \operatorname{colim}_k \pi_{n+k}(S^k) = 0$. So $x = 0$.
Otherwise, we have that x is torsion. Then, $h(x) \in MU_\bullet(\mathbb{S})$ is also torsion. But, by using A.1.11

$$MU_\bullet(\mathbb{S}) = \pi_\bullet(MU) \cong \mathbb{Z}[b_1, \cdots b_n, \cdots]$$

but it is a torsion free ring. Thus, $h(x) = 0 \Rightarrow x$ nilpotent.

□ *A.2.3*

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