Lollipops on unknown degree level structures

Max Duparc

Swissogeny Day 4 : 26th November 2025

Introduction

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1. There exists conditional attacks on unknown degree level structure.
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Figure: Today's weapon

Level Structures Isogeny Problem

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Let $\phi : E \to E'$ of degree d. $E[N] = \langle P, Q \rangle$, with N smooth[a]. Let $\Gamma \subset GL_2(\mathbb{Z}_N)$, with $\gamma \in_{\$} \Gamma$:

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- The Γ-SSI$_d$ problem:

$$d, \begin{pmatrix} P \\ Q \end{pmatrix} \text{ and } \begin{pmatrix} S \\ T \end{pmatrix} = \gamma \cdot \phi \begin{pmatrix} P \\ Q \end{pmatrix} \xrightarrow{\text{Compute}} \phi$$

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- Defines a hierarchy.

Level Structures Ladder

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\begin{array}{ccc}
\mathrm{GL}_2(\mathbb{Z}_N)\text{-SSI}_d & \longrightarrow & \mathrm{GL}_2(\mathbb{Z}_N)\text{-SSI} \\
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Figure: Level structure ladder

- Going $\nearrow$ increases the difficulty.

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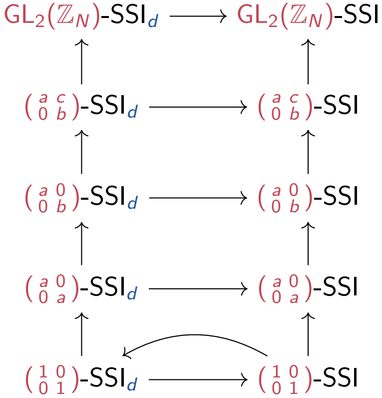

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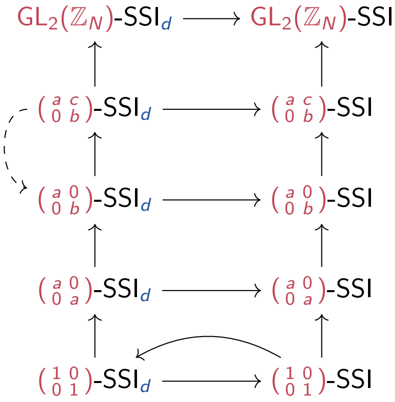

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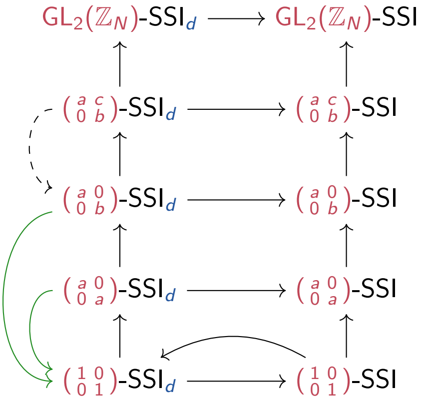

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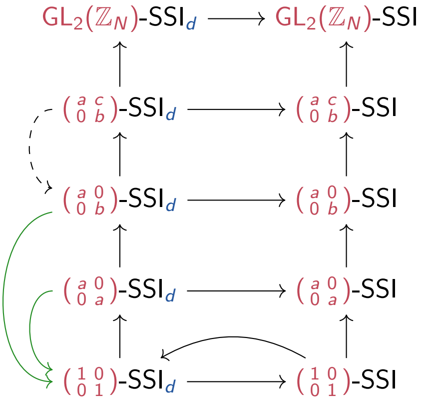

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  - [DFP24]: $\dashrightarrow$
  - [CV23]: $\longrightarrow$

- Goal: generalise $\longrightarrow$ to the unknown degree setting.

[CV23] Generalised lollipop attack

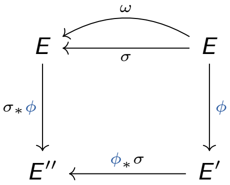

Figure: Generalised lollipop diagram

Generalised lollipop

Let $\omega, \sigma \in \mathrm{End}(E)$ with $\phi_*\sigma$ computable and $\forall \gamma \in \Gamma, (\widehat{\sigma} \circ \omega)\left(\gamma \cdot \binom{P}{Q}\right) = \gamma \cdot \left(\widehat{\sigma} \circ \omega\binom{P}{Q}\right) = \gamma \cdot \mathbf{M}\binom{P}{Q}$.

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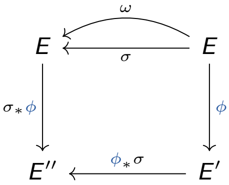

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We define $\psi = \sigma_* \phi \circ \omega \circ \widehat{\phi} : E' \to E''$ and have that

$$[\deg(\sigma)] \cdot \psi \begin{pmatrix} S \\ T \end{pmatrix} = [d] \cdot \mathbf{M} \cdot \phi_* \sigma \begin{pmatrix} S \\ T \end{pmatrix}$$

Downgrading the level structure

Key Observation

In the unknown degree setting, the generalised lollipop still downgrades the level structure.

$$[d^{-1}] \cdot \psi \begin{pmatrix} S \\ T \end{pmatrix} = [\deg(\sigma)^{-1}] \cdot \mathbf{M} \cdot \phi_* \sigma \begin{pmatrix} S \\ T \end{pmatrix}$$

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$$\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\text{-SSI}(\phi) \xrightarrow{\textit{reduction}^*} \begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\text{-SSI}(\psi)$$

Note: reduction is not perfect and eats part of ϕ oriented by $\widehat{\sigma} \circ \omega$.

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- How hard is $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI ?

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Erased degree level structure

Definition: $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI problem

$\psi : E' \to E''$ is a *cyclic* isogeny of degree d^2, and let $E'[N] = \langle S, T \rangle$ be a basis of $E'[N]$. We define the *erased degree level structure* as:

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$$e_N \left([d^{-1}]\psi(S), [d^{-1}]\psi(T) \right) = e_N(S, T)$$

  - It is an isogeny that cannot be interpolated.

$$[d^{-1}]\psi \text{ is an isogeny of degree } (1 + k_{N,d} N)^2 \gg N^2$$

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- $\mathcal{D}$ has **small support** if $|\text{supp}(\mathcal{D})| = \text{poly}(\lambda) \implies \text{lcm}(\mathcal{D}) = \exp(\lambda)$

Attacking the erased degree level structure

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For $d \in \mathcal{D}$ small supp and for N big-enough

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$$[\mathrm{lcm}(\mathcal{D})/d]\psi(E[N]) \xrightarrow{\texttt{IsogenyDiv}} [d/d]\psi(E[N])$$

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  - Can be used constructively.
- For POKE et al. [BM25, KHKL25], no attacks (yet). (As $\mathrm{lcm}(\mathcal{D}) \geq 2^{\vartheta(2^{\lambda})}$).
  - Their security comes more from $\mathcal{D}$ than from Γ.

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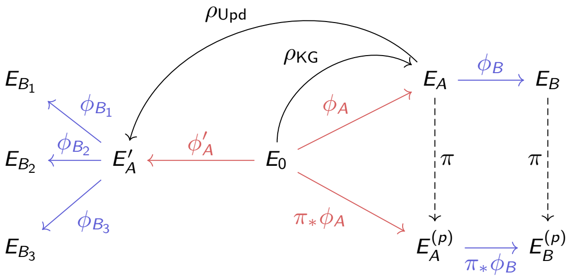
- Can be applied to construct a more efficient SILBE UPKE [DFV24].
  + Base prime p about 2.7x smaller.
  + Just need $(3,3)$ and $(3,3,3,3)$ HD-isogenies.
  + Should provide a 2^{32}x speed-up on original.
  - but p still 4700 bits for $\lambda = 128$.



Simplified overview of SILBE

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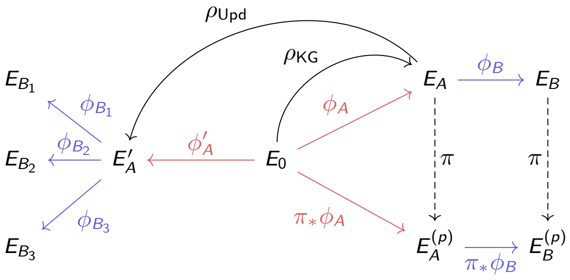
▶ Giant step in the direction of efficient UPKE.
  ▶ Work is still needed.



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Is $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI more profound ?

Level structure as partial maps

Let $\mathcal{SS}$ be the supersingular category. Assume $N = \ell^e$.

$$\eta : \mathrm{Hom}(E_1, E_2) \rightharpoonup \mathrm{Hom}(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell^e}$$
$$\eta(\phi) = \phi \otimes \sqrt{\deg(\phi)}$$

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$$\eta : \mathrm{Hom}(E_1, E_2) \rightharpoonup \mathrm{Hom}(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell^e}$$
$$\eta(\phi) = \phi \otimes \sqrt{\deg(\phi)}$$

$$\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\text{-SSI} \iff \text{compute preimage of } \eta$$

Is $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI more profound ?

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- Can we study $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI using algebraic homology ?

Conclusion

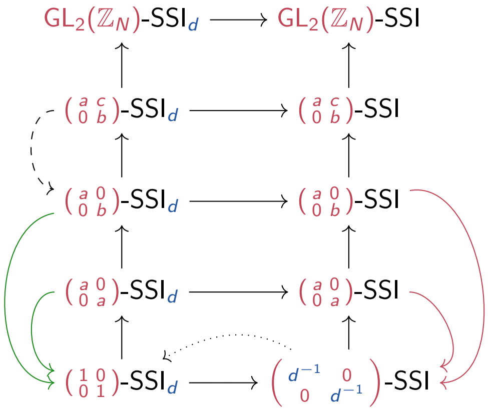

Figure: NEW Level structure ladder

Lollipops are boomerang !

Happy to discuss your comments & questions !

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