Into the rabbit hole: Computing HD isogenies through graph theory

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Current state of isogenies

- HD isogenies are everywhere:
  - dim 2: SQIsign (& var.), PRISM, POKE (& var.), DeuringVRF, ...
  - dim 4: SQIsignHD, (qt-)Pegasis, $\otimes$-MIKE, ...
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- Big limiting factor of isogeny based cryptography in near futur !

History rhymes

"The security of cryptosystems based on elliptic curves is not well understood, due in large part to the abstruse nature of elliptic curves. Few cryptographers understand elliptic curves [...] Over time, this may change, but for now trying to get an evaluation of the security of an elliptic-curve cryptosystem is a bit like trying to get an evaluation of some recently discovered Chaldean poetry."

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1. Introduce simplified theory to HD isogenies[a] !
2. Use it to gain speed-up and flexibility !
3. Explore the rabbit hole of new possibilities.

[a] Focus on 2-isogenies.

On the shoulders of giants

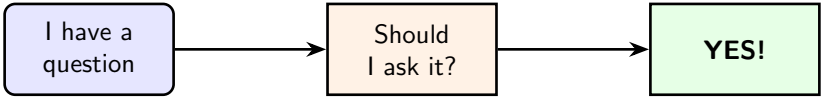
Talk based on the following works:

- [Dup25] *Superglue: Fast formulae for (2,2)-gluing isogenies*, single author, ASIACRYPT 2025.
- [DD26] *Chasing Rabbits Through Hypercubes: Better algorithms for higher dimensional 2-isogeny computations*, with Pierrick Dartois, preprint, 2026.
- $\otimes$*-MIKE for mere mortals (working title)*, with the MIKE-team *(in preparation)*.

Part is a palimpsest of the following work:

- [Dar25] *Fast computation of higher dimensional isogenies for cryptographic applications*, by Pierrick Dartois, PhD Thesis, 2025

This is an IOP



Always remember [Sha92]

$$\mathsf{IP} = \mathsf{PSPACE}$$

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Abelian varieties

$$E$$

- Abelian varieties of dim 1.
- $E[N] \cong \mathbb{Z}_N^2$

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- Surjective morphism $\implies |\ker(f)| < \infty$.
- Compatible with group structures.
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Isogenies

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- Compatible with group structures.
- $\ker(f)$ max. isotropic ($e^A|_{\ker(f)^2} \equiv 1$).

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- dual $\widetilde{f} : (B, e^B) \to (A, e^A)$ with $\ker(\widetilde{f}) = f\,(A[\deg(f)])$.

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- Using $\widehat{\mathbb{Z}_N^g} = \{\chi : \mathbb{Z}_N^g \longrightarrow \mu_N\}$ the dual of $\mathbb{Z}_N^g$[a], let:

$$\epsilon : \left(\mathbb{Z}_N^g \times \widehat{\mathbb{Z}_N^g}\right)^2 \longrightarrow \mu_N, \quad \text{s.t.} \quad \epsilon\big((x_1,\chi_1),(x_2,\chi_2)\big) = \chi_2(x_1)\cdot\chi_1(x_2)^{-1}$$

[a] $\chi_i(-) \rightsquigarrow \zeta^{\langle x_i|-\rangle}$ for ζ a primitive N-root of unity and $x_i \in \mathbb{Z}_N^g$.

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- **Symplectic representation** of $A[N]$:

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$$\mu : (A[N], e_A) \xrightarrow{\cong} (\mathbb{Z}_N^g \times \widehat{\mathbb{Z}_N^g}, \epsilon)$$

- Equivalent to **symplectic basis** $\mathcal{B}_\mu := \{S_1, \dots, S_g, T_1, \dots, T_g\}$:

$$e_A(S_i, S_j) = e_A(T_i, T_j) = 1, \quad e_A(S_i, T_j) = \zeta^{\delta_{i,j}}$$

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Definition

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$$\forall P \in A, Q \in A[N] \text{ with } \mu_A(Q) = (x_Q, \chi_Q) : \quad \theta_j^A(P+Q) = \chi_Q(j)^{-1}\theta_{j+x_Q}^A(P)$$

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- How do we construct a theta structure ?

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Question: Can you spot the problem ?

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- $\mathfrak{t}_{T_j}$ are projective, so addition is not well defined !

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     $$\mathfrak{g}_{X_i'} := \pm \mathfrak{t}_{X_i}$$

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[Mum66] Mumford theorem (simplified)

Symplectic basis of $A[4] \implies \Theta^A$ theta structures (of level 2)

Example: Montgomery curves

- On Kummer line $E_{/\pm 1}$:

  $\mathfrak{g}_P \cdot \mathfrak{g}_Q = \pm \mathbf{i}\mathfrak{g}_{P+Q} = -\mathfrak{g}_Q \cdot \mathfrak{g}_P$

[1] $u = a^2 + b^2$, $v = a^2 - b^2$

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$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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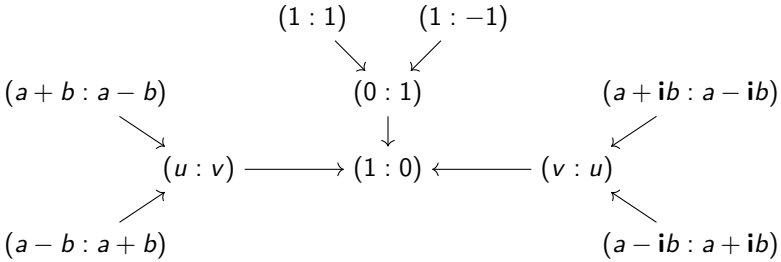

Figure: Structure of $E[4]_{/\pm 1}$ on Montgomery curves[1]

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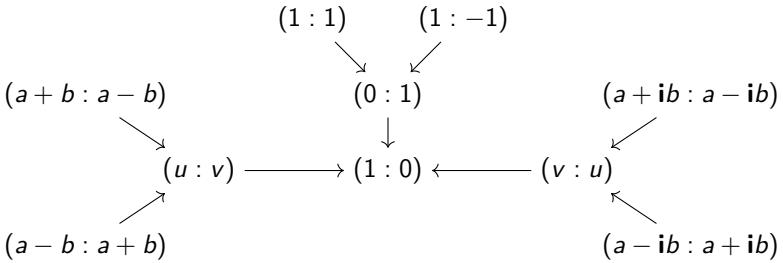

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Symmetric elements on Montgomery curves

$$\mathfrak{g}_{(1:\pm 1)} = \pm X \qquad \mathfrak{g}_{(a\pm b:a\mp b)} = \pm \frac{1}{2ab}\left(uZ - \mathbf{i}vY\right) \qquad \mathfrak{g}_{(a\pm \mathbf{i}b:a\mp \mathbf{i}b)} = \mp \frac{1}{2ab}\left(\mathbf{i}vZ + uY\right)$$

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$$\mathfrak{g}_{S_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \mathfrak{g}_{S_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathfrak{g}_{T_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{g}_{T_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Example: Theta model

Translation in the theta model

$$\mathfrak{g}_P \iff \text{theta action over } \Theta^A$$

$$\mathfrak{g}_{S_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \mathfrak{g}_{S_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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Theta model $\iff$ Fixing canonical form of $\mathfrak{g}_P$

Duality on theta structures

Duality on theta structures

$$A[4] = \{S_1, \dots, S_g, T_1, \dots, T_g\}$$

$$A[4] = \{T_1, \dots, T_g, -S_1, \dots, -S_g\}$$

Duality on theta structures

$$A[4] = \{S_1, \dots, S_g, T_1, \dots, T_g\} \quad \xleftrightarrow{\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}} \quad A[4] = \{T_1, \dots, T_g, -S_1, \dots, -S_g\}$$

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$$\Downarrow \qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$$

$$\theta^A : A \longrightarrow \mathbb{P}^{2^{g-1}} \qquad\qquad \widetilde{\theta}^A : A \longrightarrow \mathbb{P}^{2^{g-1}}$$

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$$\mathcal{H} : (x_i)_{i \in \mathbb{Z}_2^g} \longrightarrow \left(\sum_{i \in \mathbb{Z}_2^g} (-1)^{\langle j | i \rangle} x_i \right)_{j \in \mathbb{Z}_2^g}$$

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Notation:

$$(x_i)_i \odot (y_i)_i = (x_i y_i)_i \qquad (x_i)_i^{\odot n} = (x_i^n)_i \text{ for } n \in \mathbb{Z}.$$

Isogeny Formula

$$(A, e_A) \xrightarrow{f \text{ of degree } 2} (B, e_B)$$

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Isogeny Formula

$$\mathcal{H}\left(\theta^A(P+Q) \odot \theta^A(P-Q)\right) = \widetilde{\theta}^B(f(P)) \odot \widetilde{\theta}^B(f(Q))$$

Arithmetic on theta coordinates

Addition Formula

$$\mathcal{H}\left(\theta^A(P+Q)\odot\theta^A(P-Q)\right)\odot\mathcal{H}\left(\theta^A(0)^{\odot 2}\right)=\left(\widetilde{\theta}^B(f(P))\odot\widetilde{\theta}^B(f(Q))\right)\odot\left(\widetilde{\theta}^B(0)^{\odot 2}\right)$$

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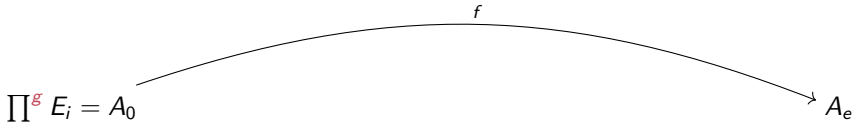
- [Gau07] This is a differential arithmetic on A.
  - simple !
  - fast !

Table of Contents

Computing Isogenies



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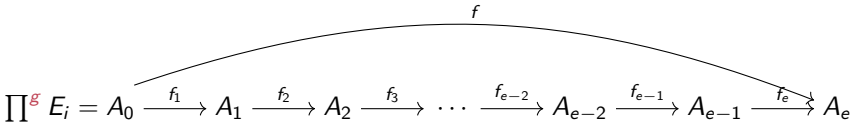

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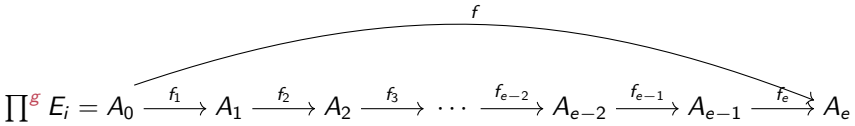

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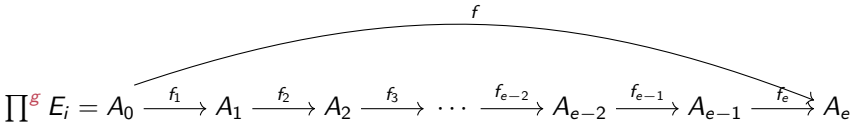

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$$\theta^A(P) \xrightarrow{S} \theta^A(P)^{\odot 2} \xrightarrow{\mathcal{H}} \mathcal{H}(\theta^A(P)^{\odot 2}) \xrightarrow{\odot \widetilde{\theta}^B(0_B)^{\odot -1}} \widetilde{\theta}^B(f(P)) \xrightarrow{\mathcal{H}} \theta^B(f(P)).$$

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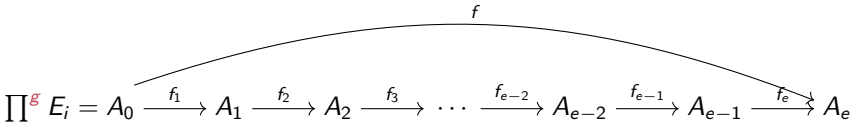

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- Bottleneck: How do we compute $\widetilde{\theta}^B(0_B)^{\odot -1}$?
  - Not trivial !

Computing the inverse dual theta null point

$$\widetilde{\theta}^B(0)^{\odot -1} \quad \rightsquigarrow \quad \theta^B(0)$$

Computing the inverse dual theta null point

$$A[8] = \langle S_1, \cdots, S_g, T_1, \cdots, T_g \rangle$$

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Goal: [DMPR24, Dar24] Solve the equation systems

$$\left\{ \widetilde{\theta}^B_i(f(T_{\mathbf{k}})) \cdot \widetilde{\theta}^B_i(0) = \alpha_{i,\mathbf{k}} \,\middle|\, \mathbf{k} \in S \subseteq \mathbb{Z}_2^g, i \in [2^g] \right\}$$

IIP

Let:

- $S \subseteq \mathbb{Z}_2^g$ be the *support*.
- $(\pi_{i,j})_{i \in \mathbb{Z}_2^g, j \in S}$ be such:

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Inverse Interpolation Problem

$$(\pi_{i,j})_{i \in \mathbb{Z}_2^g, j \in S} \xrightarrow{Compute^a} \mathbf{x}^{\odot -1} := (x_i^{-1})_{i \in \mathbb{Z}_2^g}$$

$$\text{such that } \forall i \in \mathbb{Z}_2^g, j \in S, \quad x_i^{-1} \cdot \pi_{i,j} = x_{i+j}^{-1} \cdot \pi_{i+j,j},$$

a: up to projective factor

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[DD26] IIP theorem (informal)

It's just graph topology !

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Hypercube Inverse Interpolation Problem

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It's just graph topology !

HIIP as Graph Theory

$\pi_{i,j} \leftrightsquigarrow G := (V, E)$ codomain graph

- x_i are vertices.

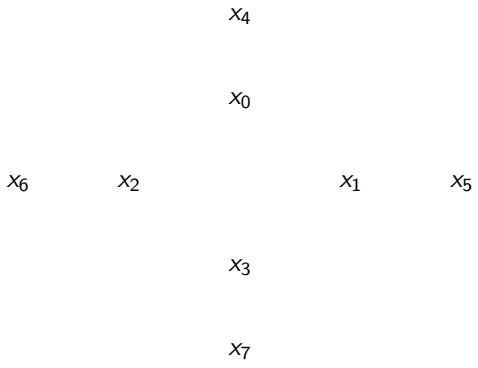

Figure: Codomain Graph G for $g = 3$, $S = \{\quad\}$

[1] assuming $\pi_{i,i+k}, \pi_{i,i+k} \neq 0$

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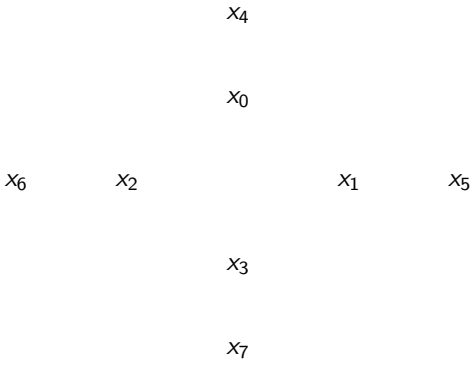

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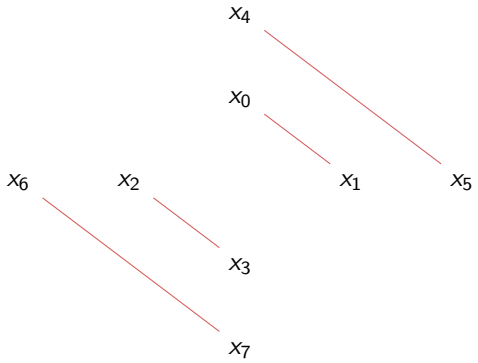

Figure: Codomain Graph G for $g = 3$, $S = \{1 \quad \}$

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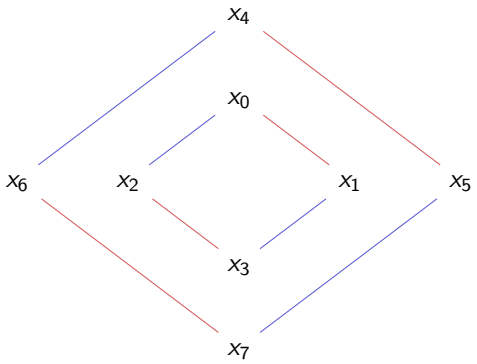

Figure: Codomain Graph G for $g = 3$, $S = \{1, 2 \quad \}$

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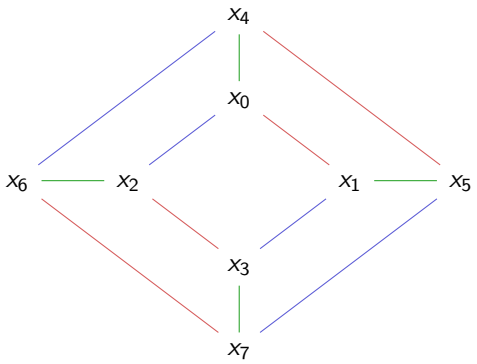

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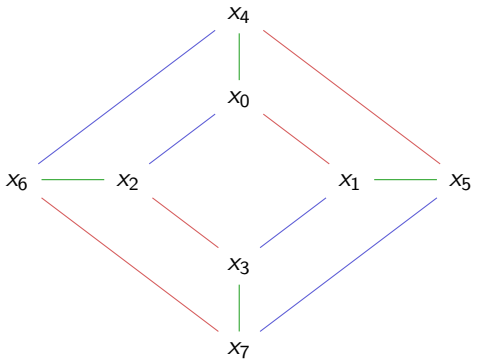
HIIP Theorem I

HIIP has a unique solution

$\iff$ G is connected



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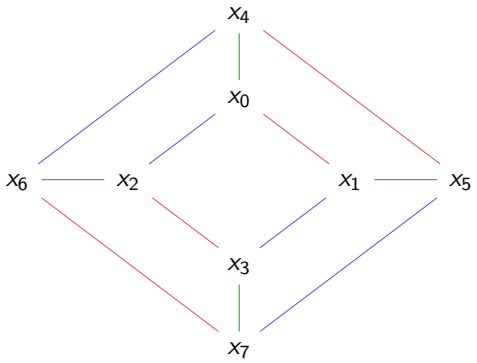
- We know when the codomain is computable !



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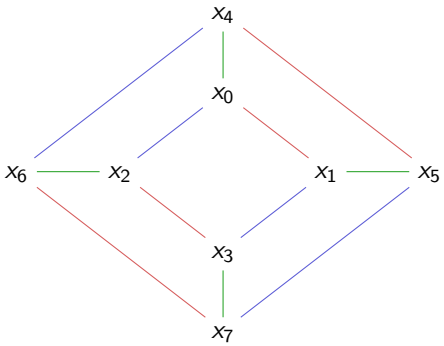

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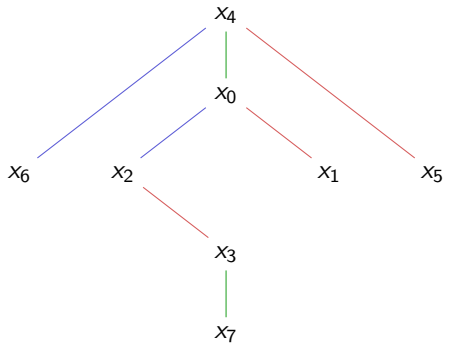

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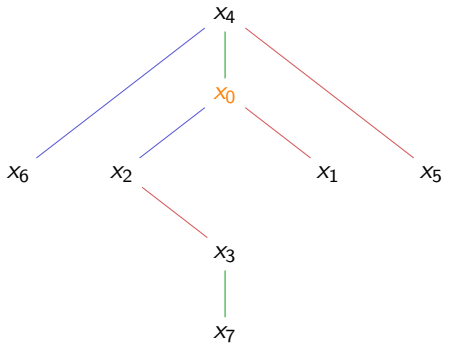

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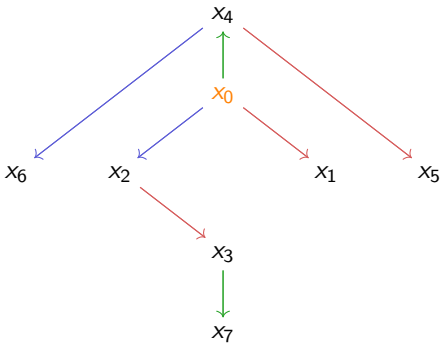

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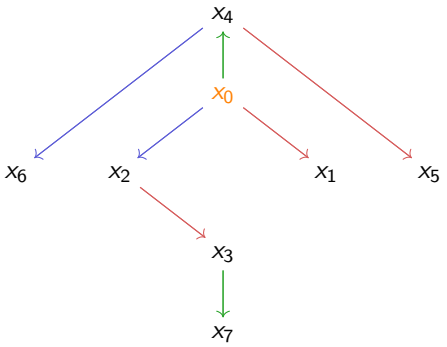

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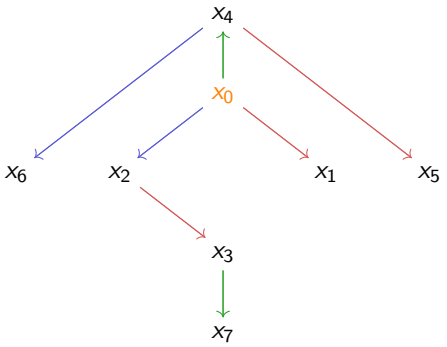

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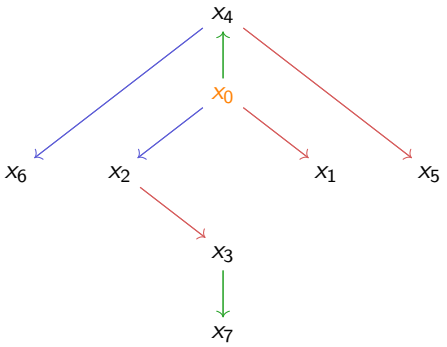

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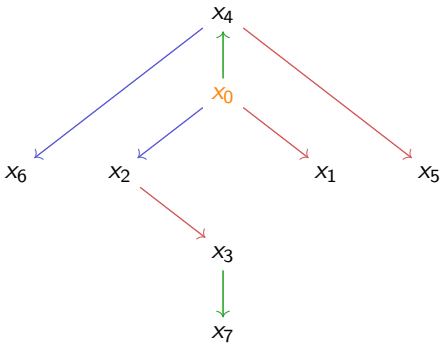
- Need good spanning tree of Hypercubes.



The case for Hamiltonian paths

- Introducing: $\mathcal{G}$ Gray paths on Hypercubes H_g.

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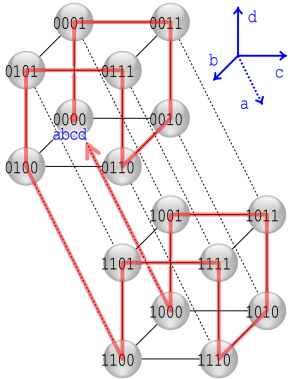

Figure: Gray code in a Tesseract

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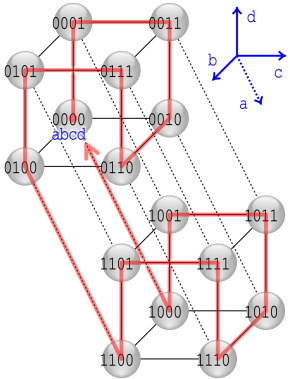
with $s_i := \eta(i+1) - \eta(i)$.



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Codomain computation in its full glory

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- Computing HIIP can be done in $(3 \cdot 2^g - 6)\mathbf{M}$!
- Can be highly parallelised !

Retrospectively trivial

Algorithm 1: Solution to the HIIP

Data: HIIP input data, including a subset $S \subseteq (\mathbb{Z}/2\mathbb{Z})^g$, a graph $G := ((\mathbb{Z}/2\mathbb{Z})^g, E)$ and matrix $(\pi_{\mathbf{i},\mathbf{j}})_{\mathbf{i}\in(\mathbb{Z}/2\mathbb{Z})^g,\mathbf{j}\in S}$. A Hamiltonian path $\eta : [\![0, 2^g - 1]\!] \xrightarrow{\sim} (\mathbb{Z}/2\mathbb{Z})^g$ in G.

Result: A solution $(x_{\eta(i)}^{\odot -1})_{0\leq i\leq 2^g-1}$ to the HIIP defined by the input data (up to reordering by η).

$\rho_L(0) \leftarrow 1$;
$\rho_L(1) \leftarrow \pi_{\eta(1),\mathbf{s}_1}$;
$\rho_R(2^g - 1) \leftarrow 1$;
$\rho_R(2^g - 2) \leftarrow \pi_{\eta(2^g-1),\mathbf{s}_{2^g-2}}$;
for $i \in [\![2, 2^g - 1]\!]$ **do**
  $\rho_L(i) \leftarrow \rho_L(i-1) \cdot \pi_{\eta(i),\mathbf{s}_i}$;
  $\rho_R(2^g - 1 - i) \leftarrow \rho_L(2^g - i) \cdot \pi_{\eta(2^g-i),\mathbf{s}_{2^g-1-i}}$;
end
$x_{\eta(0)}^{-1} \leftarrow \rho_R(0)$;
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for $i \in [\![1, 2^g - 2]\!]$ **do**
  $x_{\eta(i)}^{-1} \leftarrow \rho_L(i) \cdot \rho_R(i)$;
end
return $\left(x_{\eta(i)}^{-1}\right)_{0\leq i\leq 2^g-1}$; // Total cost: $(3 \cdot 2^g - 6)\mathbf{M}$

Generic isogeny: computing cost

	dim. g	dim. 4
Old [Dar25]	$(6 \cdot 2^g - 9)\mathbf{M} + g2^g\mathbf{S} + g^2 2^g \mathbf{a}$	$87\mathbf{M} + 64\mathbf{S} + 256\mathbf{a}$
New [DD26]	$3(2^g - 2)\mathbf{M} + g2^g\mathbf{S} + g^2 2^g \mathbf{a}$	$42\mathbf{M} + 64\mathbf{S} + 256\mathbf{a}$

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Graph theory gives us formulae that are:

- Simple to implement !
- Constant time !
- *Very flexible !*

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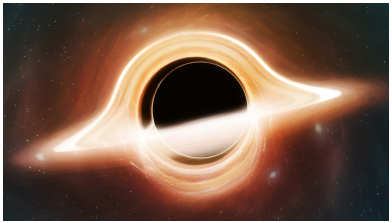

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Gluing is:

- A rabbit-hole of technicality.
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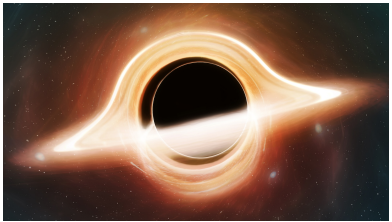

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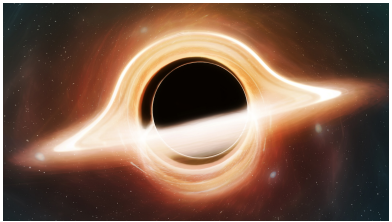
- HIIP is black-hole compatible !



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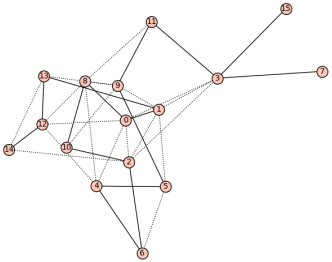

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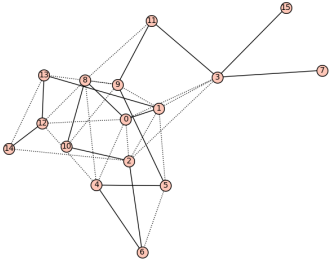

Figure: Rabbit picture

- Computing a rabbits take $(3 \cdot 2^g - 5)\mathbf{M}$
- Using other (many) technical tricks, made constant time gluing for qt-Pegasis.

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Example: Isogenies between fourfold of Weil-type $(2,2)$

We have $\iota \in \mathrm{Aut}(A), \iota^2 = -1$

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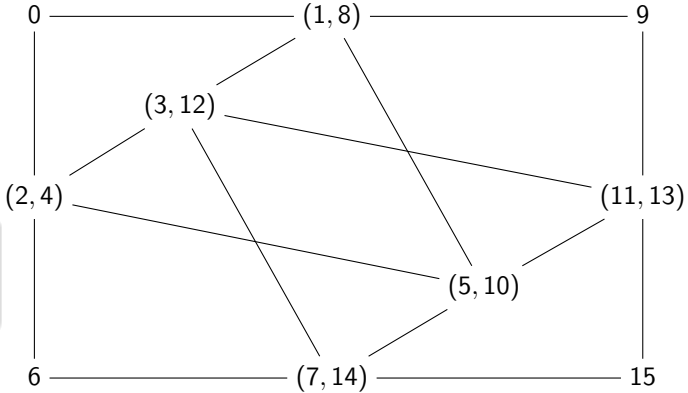
- Computable using only 2 points!



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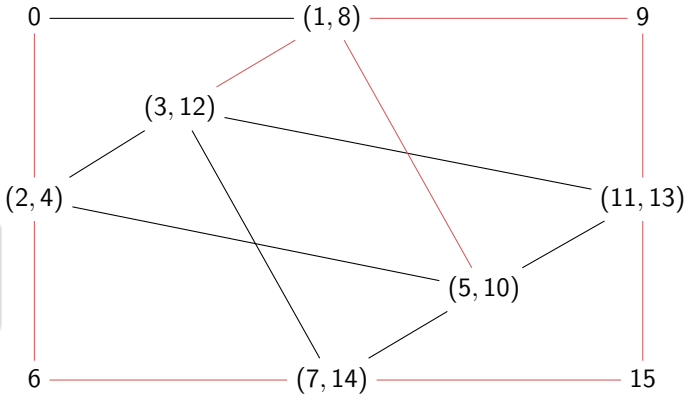

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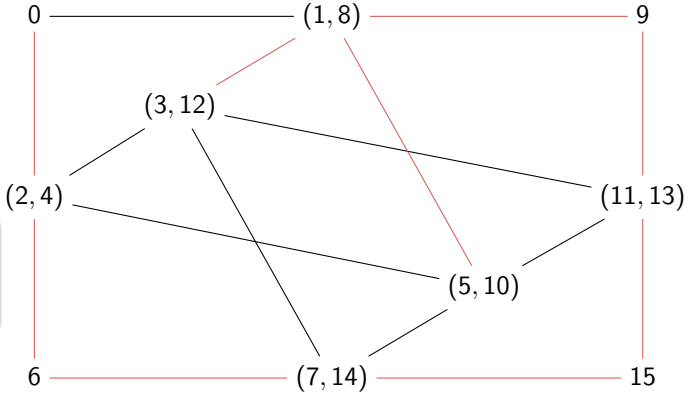

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Corollary

$\otimes$-MIKE is fast !

Implementation of qt-Pegasis

$\lceil \log_2(p) \rceil$	p	ARM (Mcycles)	X86 (Mcycles)	Timing (ms)
505	$27 \cdot 2^{500} - 1$	146.95	274.38	23.94
1008	$15 \cdot 2^{1004} - 1$	871.34	1627.05	146.19
1525	$5 \cdot 2^{1522} - 1$	3163.05	5208.81	543.01
2017	$5 \cdot 2^{2014} - 1$	7041.82	11192.45	1222.96
4030	$45 \cdot 2^{4024} - 1$	55228.89	83757.68	10654.93

Table: Performance of the C-implementation of 4-dimensional chain of qt-Pegasis in CPU Mcycles. Results are the average of 100 benchmark runs.

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1. Dim 4 chains are constructively practical!
2. We have all the tools for generic dimension!
3. It has never been easier to join the train!
4. Gluing will provide you *endless* fun!

Thanks for listening !

Beware of rabbits !

Ad break

Workshop on Abelian varieties and Applications to Post-Quantum Cryptography

16 - 20 November 2026
Bernoulli center, EPFL

More info on: https://bernoulli-isogeny.github.io/index.html

Figure: EPFL (not in November)

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