The rabbit hole of theta stucture and their constructive applications

Max Duparc

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- Thanks to **theta structures**.
  - Let's explore this rabbit hole !

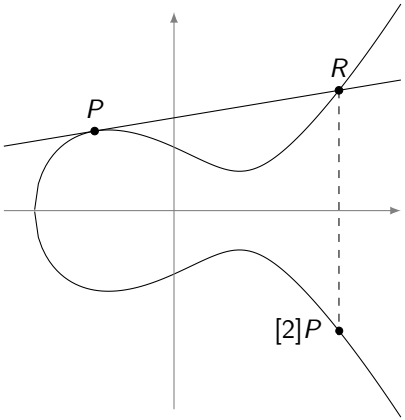
Elliptic curves

Let $\mathbb{F}_q$ be our basis field.

- *Weierstrass equations*:

$$E : y^2 = x^3 + Ax + B$$

with $4A^3 + 27B^2 \neq 0$.



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- *Weierstrass equations*:

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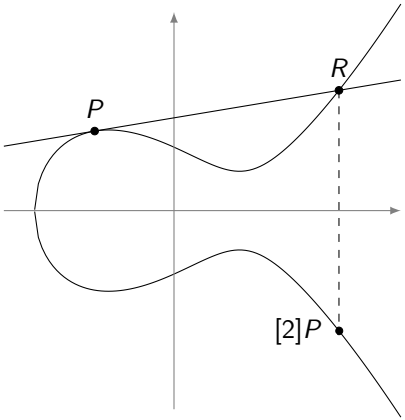
- *j-invariant*:

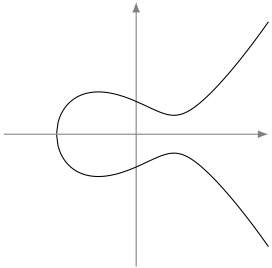
$$j(E) = 1728\frac{4A^3}{4A^3 + 27B^2}$$

characterises isomorphism classes.

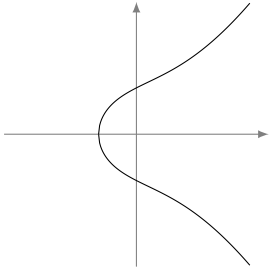


Isogenies

- Surjective group morphism between elliptic curves.



$E : y^2 = x^3 - 3x + 3$



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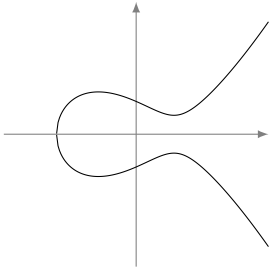

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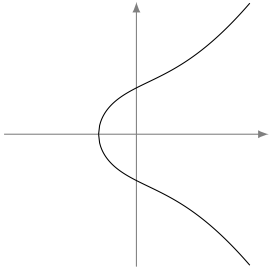

$E' : y^2 = x^3 + 5x + 6$

$$\phi : (x, y) \to \left(\frac{x^2 + 6x + 1}{x - 7}, \frac{x^2 - x - 4}{(x - 7)^2} y \right) \text{ in } \mathbb{F}_{13}$$

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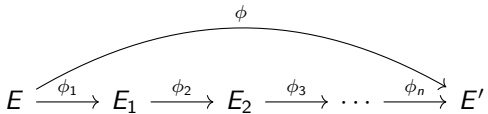
- Let $\deg(\phi) = \ell^n$. Can compute it in $O\big(n \log(n) \ell\big)$



Abelian Varieties

E

- Abelian varieties of dim 1.
- $E[N] \cong \mathbb{Z}_N^2$
- isogeny $\phi : E \to E'$ with finite kernel.
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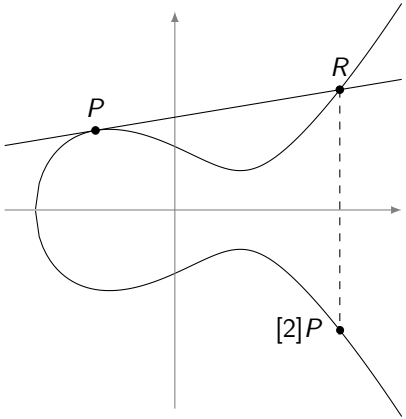

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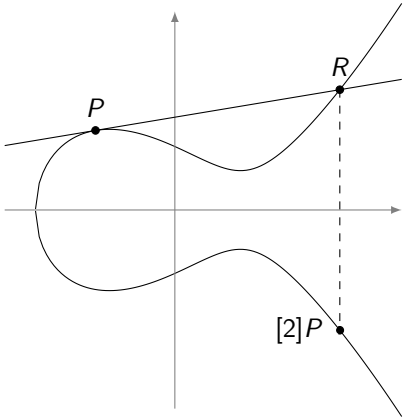

Abelian Varieties

$$(A, e^A)$$

- Abelian varieties of dim g.
- $A[N] \cong \mathbb{Z}_N^{2g}$
- isogeny $\phi : A \to A'$ with finite max isotropic* kernel. $(e^A|_{\ker(f)^2} \equiv 1)$.
- $\deg(\phi) = |\ker(\phi)|^g$.
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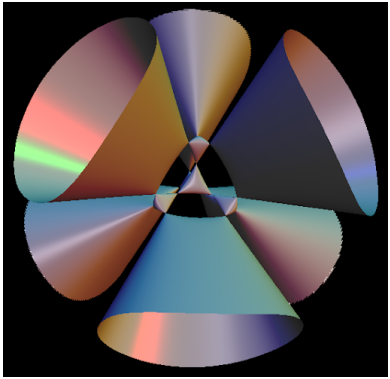

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Isogeny interpolation

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- We need fast arithmetic and isogenies on Abelian varieties.

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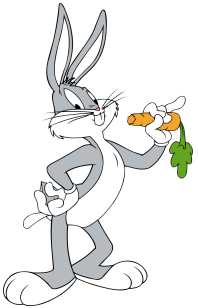

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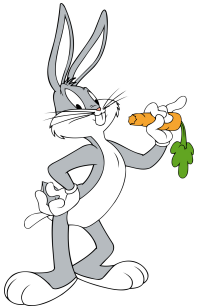

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  - [Dar24]: Generalisation of [DMPR24].
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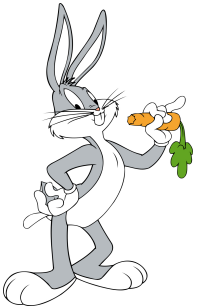

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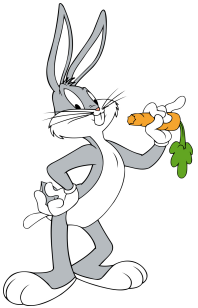

We now have all tools to compute fast HD isogenies (of degree 2^ℓ).

Symplectic form

- Theta structures are old [Mum66].
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Symplectic representation

- Using $\widehat{\mathbb{Z}_N^g} = \{\chi : \mathbb{Z}_N^g \longrightarrow \mu_N\}$ the dual of $\mathbb{Z}_N^g$[a], let:

$$\epsilon : \left(\mathbb{Z}_N^g \times \widehat{\mathbb{Z}_N^g}\right)^2 \longrightarrow \mu_N, \quad \text{s.t.} \quad \epsilon\big((x_1,\chi_1),(x_2,\chi_2)\big) = \chi_2(x_1)\cdot\chi_1(x_2)^{-1}$$

[a] $\chi_i(-) \leftrightsquigarrow \zeta^{\langle x_i|-\rangle}$ for ζ a primitive N-root of unity and $x_i \in \mathbb{Z}_N^g$.

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- **Symplectic representation** of $A[N]$:

$$\mu : (A[N], e_A) \xrightarrow{\cong} (\mathbb{Z}_N^g \times \widehat{\mathbb{Z}_N^g}, \epsilon)$$

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- Equivalent to **symplectic basis** $\mathcal{B}_\mu := \{S_1, \dots, S_g, T_1, \dots, T_g\}$:

$$e_A(S_i, S_j) = e_A(T_i, T_j) = 1, \quad e_A(S_i, T_j) = \zeta^{\delta_{i,j}}$$

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satisfying the *theta group action relation*:

$$\forall P \in A, Q \in A[N] \text{ with } \mu_A(Q) = (x_Q, \chi_Q) : \quad \theta_j^A(P+Q) = \chi_Q(j)^{-1}\theta_{j+x_Q}^A(P)$$

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- Mainly interested in $N = 2$:
  - $\theta^A(0)$ characterises A (up to isomorphism).
  - It is *symmetric*: (i.e. $\theta^A(-P) = \theta^A(P)$).

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$$P \in A[4] \implies \theta^A(P) \text{ is fixed by the theta action of } [2]P$$

	0	T_1	T_2	$T_1 + T_2$
0	$-$	$(x:0:y:0)$	$(x:y:0:0)$	$(x:0:0:y)$
S_1	$(x:x:y:y)$	$(x:\mathbf{i}x:y:\mathbf{i}y)$	$(x:x:y:-y)$	$(x:\mathbf{i}x:y:-\mathbf{i}y)$
S_2	$(x:y:x:y)$	$(x:y:x:-y)$	$(x:y:\mathbf{i}x:-\mathbf{i}y)$	$(x:y:-\mathbf{i}x:\mathbf{i}y)$
$S_1 + S_2$	$(x:y:y:x)$	$(x:y:-\mathbf{i}y:\mathbf{i}x)$	$(x:y:\mathbf{i}y:\mathbf{i}x)$	$(x:y:-y:x)$

Table: Structure of $\theta^A(P)$ depending on the position of $[2]P \in A[2]$

Computation

Notation:

$$(x_i)_i \odot (y_i)_i = (x_i y_i)_i$$
$$(x_i)_i^{\odot n} = (x_i^n)_i \text{ for } n \in \mathbb{Z}.$$

$$\mathcal{H} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \text{ Hadamard transform}$$

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[Mum66]: The isogeny formula

Let $f : A \to B$ by a 2-isogeny, then:

$$\mathcal{H}\left(\theta^A(P+Q) \odot \theta^A(P-Q)\right) = \mathcal{H}\big(\theta^B(f(P))\big) \odot \mathcal{H}\big(\theta^B(f(Q))\big)$$

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- [Gau07] Gives a fast (differential) arithmetic on A.

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- Bottleneck: computing $\mathcal{H}\left(\theta^B(0_B)\right)^{\odot -1}$?

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[DD26] Theorem II

$$\text{From } \mathcal{T} \subseteq \mathcal{G} \text{ spanning tree} \implies \text{formulae for } \theta^{\odot -1}(0).$$

Computing the inverse dual theta null point

$$\mathcal{B} \subseteq \ker(f) \iff (\pi_{\mathbf{i},j})_{\mathbf{i}\in\mathbb{Z}_2^g,\mathcal{B}} \text{ system of equations} \iff \text{a graph } \mathcal{G}.$$

[DD26] Theorem I

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- For generic isogenies of dim g, $\mathcal{G}$ is a hypercube.
  - Need good spanning tree of hypercubes.

A good spanning tree on hypercube

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- Gray paths on hypercubes H_g.

$$\eta : [2^g] \xrightarrow{\cong} H_g$$

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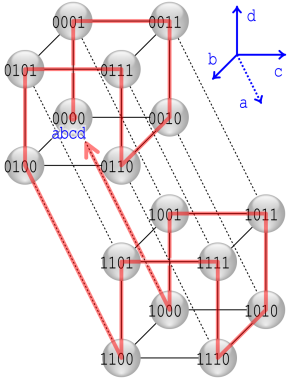

Figure: Gray code in a H_4

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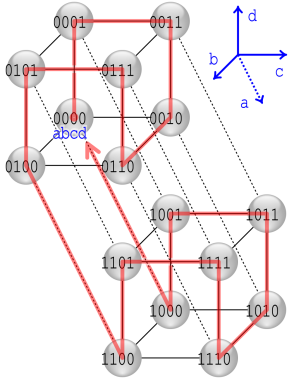
$$x^{-1}_{\eta(i)} = \left(\prod_{j=0}^{i-1} \pi_{\eta(j),s_j} \right) \cdot \left(\prod_{j=i}^{2^g-2} \pi_{\eta(j+1),s_j} \right)$$

with $s_i := \eta(i+1) - \eta(i)$.

Consequence

2x faster & constant time.



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Exotic case: Gluing

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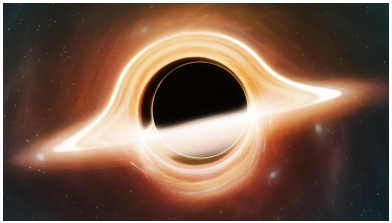

Figure: Artist view of a gluing isogeny

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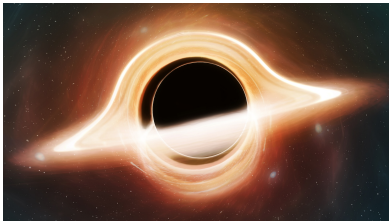

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Gluing is:

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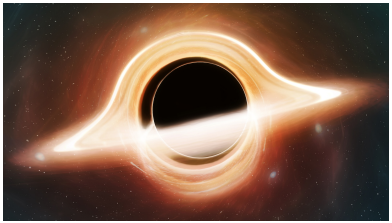
- The hardest part of the implementation.
- Computable thanks to [DD26] !



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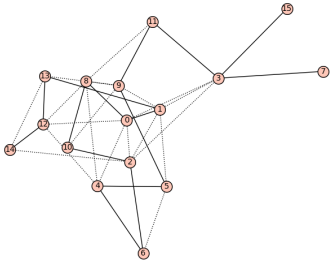

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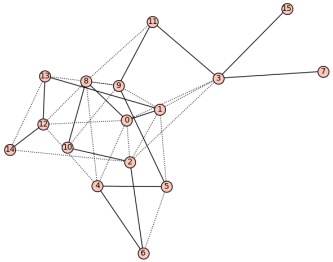

Figure: A rabbit

- Almost as fast as generic case.
- Using **many** others technical tricks, can make constant time gluing for qt-Pegasis.
  - Base of the *PQarrot* crypto-suite.

Exotic case: Special varieties

On some special Abelian varieties, theta structures have extra symmetries:

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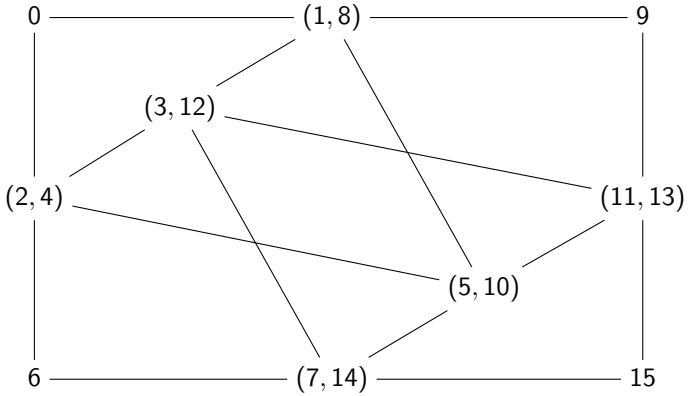

Figure: Graphs abelian fourfold of Weil type

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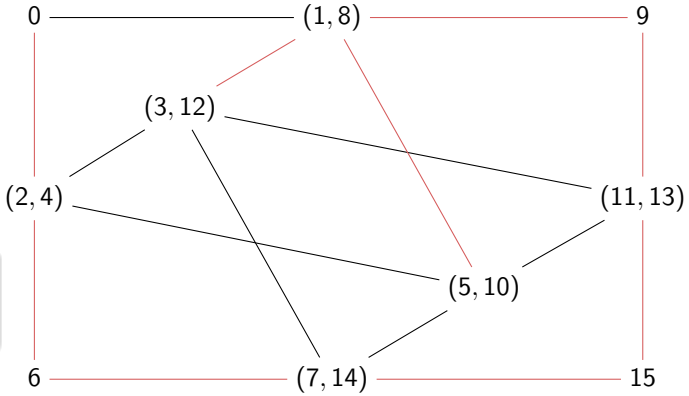
Consequence

Can compute isogenies (and arithmetic) faster !



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Implementation of dim 4 chains

$\lceil \log_2(p) \rceil$	ARM (Mcycles)	X86 (Mcycles)	Timing (ms)
505	146.95	274.38	23.94
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Timings collected on arm 64 M4 Pro, 4.5GHz.

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$\lceil \log_2(p) \rceil$	sk	pk	KeyGen (ms)	KeyAgrement (ms)
256	32 B	64 B	0.335	2.238
384	48 B	92 B	1.095	8.720
512	64 B	128 B	2.525	21.017

Table: Performance of the Rust implementation of 4D chain on Weil type curve (MIKE).

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Thanks for listening !

Beware of rabbits !

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