Lollipops on unknown degree level structures

Max Duparc

Swissogeny Day 4 : 26th November 2025

Introduction

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In this presentation

1. There exists conditional attacks on unknown degree level structure.
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2. This opens interesting research directions.

Figure: Today's weapon

Level Structures Isogeny Problem

Definition: Level Structures Isogeny Problem [DFP24]

Let $\phi : E \to E'$ of degree d. $E[N] = \langle P, Q \rangle$, with N smooth[a]. Let $\Gamma \subset \mathrm{GL}_2(\mathbb{Z}_N)$, with $\gamma \in_{\$} \Gamma$:

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- The Γ-SSI_d problem:

$$d, \begin{pmatrix} P \\ Q \end{pmatrix} \text{ and } \begin{pmatrix} S \\ T \end{pmatrix} = \gamma \cdot \phi \begin{pmatrix} P \\ Q \end{pmatrix} \xrightarrow{\text{Compute}} \phi$$

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- Defines a hierarchy.

Level Structures Ladder

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- We are searching for $\swarrow$ transformations.
  - [DFP24]: $\dashrightarrow$
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- Goal: generalise $\longrightarrow$ to the unknown degree setting.

[CV23] Generalised lollipop attack

$$\begin{tikzcd}
E \arrow[d, "\sigma_*\phi"'] & E \arrow[l, "\sigma"] \arrow[l, bend right, "\omega"'] \arrow[d, "\phi"] \\
E'' & E' \arrow[l, "\phi_*\sigma"']
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Figure: Generalised lollipop diagram

Generalised lollipop

Let $\omega, \sigma \in \mathrm{End}(E)$ with $\phi_*\sigma$ computable and $\forall \gamma \in \Gamma, (\widehat{\sigma} \circ \omega)\left(\gamma \cdot \binom{P}{Q}\right) = \gamma \cdot \left(\widehat{\sigma} \circ \omega\binom{P}{Q}\right) = \gamma \cdot \mathbf{M}\binom{P}{Q}$.

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We define $\psi = \sigma_*\phi \circ \omega \circ \widehat{\phi} : E' \to E''$ and have that

$$[\deg(\sigma)] \cdot \psi \begin{pmatrix} S \\ T \end{pmatrix} = [d] \cdot \mathbf{M} \cdot \phi_*\sigma \begin{pmatrix} S \\ T \end{pmatrix}$$

Downgrading the level structure

Key Observation

In the unknown degree setting, the generalised lollipop still downgrades the level structure.

$$[d^{-1}] \cdot \psi\begin{pmatrix} S \\ T \end{pmatrix} = [\deg(\sigma)^{-1}] \cdot \mathbf{M} \cdot \phi_* \sigma \begin{pmatrix} S \\ T \end{pmatrix}$$

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Note: reduction is not perfect and eats part of ϕ oriented by $\widehat{\sigma} \circ \omega$.

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- How hard is $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI ?

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Erased degree level structure

Definition: $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI problem

$\psi : E' \to E''$ is a *cyclic* isogeny of degree d^2, and let $E'[N] = \langle S, T \rangle$ be a basis of $E'[N]$. We define the *erased degree level structure* as:

$$\begin{pmatrix} S \\ T \end{pmatrix} \text{ and } \begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix} \cdot \psi \begin{pmatrix} S \\ T \end{pmatrix} \xrightarrow{\text{Compute}} \psi$$

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$$e_N \left([d^{-1}]\psi(S), [d^{-1}]\psi(T) \right) = e_N(S, T)$$

  - It is an isogeny that cannot be interpolated.

$$[d^{-1}]\psi \text{ is an isogeny of degree } (1 + k_{N,d} N)^2 \gg N^2$$

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- $\mathcal{D}$ has **small support** if $|\text{supp}(\mathcal{D})| = \text{poly}(\lambda) \implies \text{lcm}(\mathcal{D}) = \exp(\lambda)$

Attacking the erased degree level structure

Theorem

For $d \in \mathcal{D}$ small supp and for N big-enough

$$\text{solving } \begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\text{-SSI is easy}$$

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$$[\mathrm{lcm}(\mathcal{D})/d]\psi(E[N]) \xrightarrow{\texttt{IsogenyDiv}} [d/d]\psi(E[N])$$

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Cryptanalytic consequences

By design: Rationality $\implies$ small support.

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  - Can be used constructively.
- For POKE et al. [BM25, KHKL25], no attacks (yet). (As $\mathrm{lcm}(\mathcal{D}) \geq 2^{\vartheta(2^{\lambda})}$).
  - Their security comes more from $\mathcal{D}$ than from Γ.

Construct using lollipops

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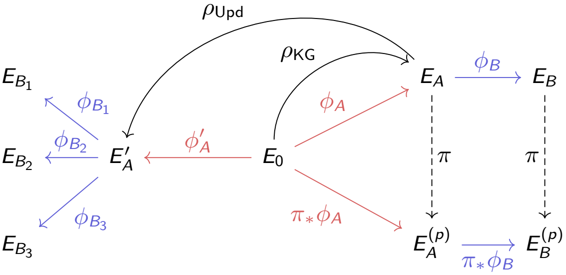
  - + Base prime p about 2.7x smaller.
  - + Just need $(3,3)$ and $(3,3,3,3)$ HD-isogenies.
  - + Should provide a 2^{32}x speed-up on original.
  - - but p still 4700 bits for $\lambda = 128$.



Simplified overview of SILBE

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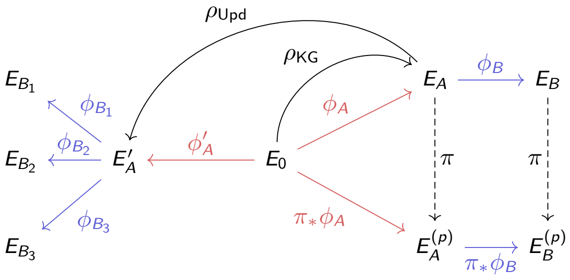
- ▶ Giant step in the direction of efficient UPKE.
  - ▶ Work is still needed.



Simplified overview of SILBE

Is $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI more profound ?

Level structure as partial maps

Let $\mathcal{SS}$ be the supersingular category. Assume $N = \ell^e$.

$$\eta : \mathrm{Hom}(E_1, E_2) \rightharpoonup \mathrm{Hom}(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell^e}$$
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- Can we study $\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}$-SSI using algebraic homology ?

Conclusion

$$
\begin{tikzcd}
\mathrm{GL}_2(\mathbb{Z}_N)\text{-SSI}_d \arrow[r] & \mathrm{GL}_2(\mathbb{Z}_N)\text{-SSI} \\
\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)\text{-SSI}_d \arrow[r] \arrow[u] \arrow[d, dashed, bend right] & \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)\text{-SSI} \arrow[u] \\
\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\text{-SSI}_d \arrow[r] \arrow[u] \arrow[dd, bend right] & \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\text{-SSI} \arrow[u] \arrow[dd, bend left] \\
\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\text{-SSI}_d \arrow[r] \arrow[u] \arrow[d, bend right] & \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\text{-SSI} \arrow[u] \arrow[d, bend left] \\
\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\text{-SSI}_d \arrow[r] \arrow[u] & \begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\text{-SSI} \arrow[u] \arrow[l, dotted, bend right]
\end{tikzcd}
$$

Figure: NEW Level structure ladder

Lollipops are boomerang !

Happy to discuss your comments & questions !

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